2-Cartesian fibrations II: Higher cofinality

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Abstract

In this work, we characterize cofinal functors of $(\infty,2)$ -categories via generalizations of the conditions of Quillen's Theorem A. In a special case, our main result recovers Joyal's well-known characterization of cofinal functors of $(\infty,1)$ -categories. As a stepping stone to the proof of this characterization, we use the theory of 2-Cartesian fibrations developed in previous work to provide an $(\infty,2)$ -categorical Grothendieck construction. Given a scaled simplicial set S we construct a 2-categorical version of Lurie's straightening-unstraightening adjunction, thereby furnishing an equivalence between the ∞ -bicategory of 2-Cartesian fibrations over S and the ∞ -bicategory of contravariant functors $S^{\mathrm{op}} \to \mathbb{B}\mathrm{icat}_{\infty}$ with values in the ∞ -bicategory of ∞ -bicategories.

Keywords: Cofinality, Quillen's Theorem A, $(\infty, 2)$ -category, colimit.

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1. Introduction

This paper is the conclusion of a long project begun in [2], with the aim of providing a characterization of cofinal functors of $(\infty, 2)$ -categories. The theory of colimits in its ∞ -bicategorical formulation [11], takes as indexing diagrams scaled simplicial sets K equipped with a collection of marked edges, which encode the level of laxness of the colimit to be taken. We will frequently denote such marked-scaled simplicial sets with a superscript K^{\dagger} . In this context, our main theorem shows how the notion of 2-categorical cofinality is controlled by conditions analogous to those of Quillen's Theorem A [26].

Theorem. Let $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ be a marking-preserving functor of ∞ -bicategories. Then the following statements are equivalent

- 1. The functor f is marked cofinal.
- 2. For every $d \in \mathbb{D}$ the functor f induces an equivalence of ∞ -categorical localizations $L_W(\mathbb{C}_{d\uparrow}^{\dagger}) \to L_W(\mathbb{D}_{d\uparrow}^{\dagger})$.

Moreover, the original characterization of cofinal functors of ∞ -categories due to Joyal (see [21, Theorem 4.1.3.1]) can be retrieved in the specific instance where all edges are marked. It is worth pointing out that this notion of marking-depending colimit (or in our terminology, marked colimit) can be shown to the equivalent to the already well-established theory of weighted colimits (see Section 1.2 for more details). In particular, our theorem can be applied to detect cofinal functors in the setting of weighted colimits. The main theorem of this paper thus provides a computational aid when dealing with the vast array of category-theoretic constructions which are typically defined in terms of weighted colimits, as for example, the 2-categorical theory of Kan extensions.

While the proof of this cofinality criterion was the main motivation for this project, the majority of this paper is devoted to constructing the toolbox needed to achieve our goal. A central role in this paper is played by one such technological tool: an $(\infty, 2)$ -categorical Grothendieck construction. Given the wide range of uses for Grothendieck constructions throughout the higher-categorical literature, we expect that the $(\infty, 2)$ -categorical variant will have applications beyond our cofinality statement. Already, an alternate $(\infty, 2)$ -categorical Grothendieck construction has been sketched — though the corresponding equivalence was not fully proven — by Gaitsgory and Rozenblyum in [13] with an eye towards applications in derived algebraic geometry. Similarly, works such as [24] have proven results conditioned on the existence of an $(\infty, 2)$ -categorical Grothendieck construction. Given the direct and explicit nature of our formulation of the $(\infty, 2)$ -categorical Grothendieck construction, it is likely that it will ease applications, in addition to providing a proof of the desired equivalence.

1.1. The Grothendieck construction

In its most basic form, for 1-categories fibred in sets, the Grothendieck construction predates Grothendieck's work on the subject (see, e.g. [25, pg. 44] for discussion). In this context, the Grothendieck construction reconstitutes the information of a functor of 1-categories

$$F \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}$$

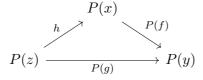
into the associated *category of elements*, a category whose objects consist of an object $c \in \mathcal{C}$, and an element $x \in F(c)$, and whose morphisms are morphisms in \mathcal{C} whose associated maps of sets preserve the chosen element.

The Grothendieck construction in its modern form emerged as a tool to study descent (see, e.g. [17]). In this case, it takes the form of an equivalence

$$\operatorname{Fib}(\mathfrak{C}) \simeq \operatorname{Fun}^{\operatorname{ps}}(\mathfrak{C}^{\operatorname{op}}, \operatorname{Cat}),$$

for any category \mathcal{C} , between the category of fibred categories over \mathcal{C} , and the category of pseudo-functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$. The underlying idea is that certain conditions on a a functor $p: \mathcal{D} \to \mathcal{C}$ mean that the fibres of p vary (pseudo-)functorially in \mathcal{C} . Indeed, the original definition of a fibred category, in [16], was what we today would call a pseudo-functor $F: \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$. More precisely, an assignment of a category $F(x) \in \mathrm{Cat}$ for every $x \in \mathcal{C}$, a functor $F(f): F(y) \to F(x)$ for every morphisms $f: x \to y$ in \mathcal{C} , and natural isomorphisms $F(g) \circ F(f) \cong F(f \circ g)$ for every composable pair of morphisms, satisfying additional coherence conditions.

The Grothendieck construction, as first exposed in [17], reformulates the data of a pseudo-functor into a *Cartesian fibration*. Given a functor $P: \mathcal{F} \to \mathcal{C}$, an morphism $f: x \to y$ in \mathcal{F} is called *Cartesian* if, for every $g: z \to y$ in \mathcal{F} , and every commutative diagram



in \mathbb{C} , there is a unique morphism $\tilde{h}: z \to x$ with $P(\tilde{h}) = h$, such that $f \circ \tilde{h} = g$. The functor P is said to be an *Cartesian fibration* if, for every $f: c \to P(y)$ in \mathbb{C} , there is a Cartesian morphism $\tilde{f}: x \to y$ in \mathcal{F} such that $P(\tilde{f}) = f$.

The equivalence between pseudo-functors $F: \mathcal{C}^{\text{op}} \to \text{Cat}$ and Cartesian fibrations over \mathcal{C} is achieved by constructing a Cartesian fibration $P: \text{El}(F) \to \mathcal{C}$ as follows:

- The objects of El(F) consist of pairs (c, x), where $c \in \mathcal{C}$, and $x \in F(c)$.
- A morphism $(f, \tilde{f}): (c, x) \to (d, y)$ consists of a morphism $f: c \to d$ in \mathcal{C} , together with a morphism $\tilde{f}: x \to F(f)(y)$ in F(x).

The Cartesian morphisms of El(F) are precisely those (f, \tilde{f}) such that \tilde{f} is an isomorphism.

1.1.1. Higher-categorical Grothendieck constructions

More recent incarnations of the Grothendieck construction have focused on ∞ -categorical variants. By their very nature, functors of $(\infty, 1)$ -categories generalize pseudo-functors of (2, 1)-categories, so that higher Grothendieck constructions now take the form of equivalences

$$\operatorname{Cart}(\mathfrak{C}) \simeq \operatorname{Fun}(\mathfrak{C}^{\operatorname{op}}, \operatorname{\mathfrak{C}at}_{\infty})$$

of ∞ -categories. This equivalence was proven by Lurie in [21], using model-categorical techniques which we adapt to make our arguments here.

The basic form of these arguments is not hard to follow. Given an ∞ -category \mathcal{C} , presented as a quasi-category, Lurie defines marked simplicial sets over \mathcal{C} to be pair (X, M_X) consisting of a simplicial set $X \in \operatorname{Set}_{\Delta}$ and a subset $M_X \subset X_1$ of marked edges containing all degenerate edges, equipped with a morphism $p: X \to \mathcal{C}$ of simplicial sets. Requiring maps to preserve these marked edges yields a category $(\operatorname{Set}_{\Delta}^+)_{/\mathcal{C}}$. Lurie then constructs a model structure on this category, the fibrant objects of which satisfy lifting properties akin to those defining 1-categorical Cartesian fibrations. In particular, the corresponding model structure on $\operatorname{Set}_{\Delta}^+ \cong (\operatorname{Set}_{\Delta}^+)_{/\Delta^0}$ models $(\infty, 1)$ -categories.

With these model structures in place, one can consider the category $(\operatorname{Set}_{\Delta}^+)^{\mathfrak{C}[\mathcal{C}]^{\operatorname{op}}}$ of simplicially-enriched functors $\mathfrak{C}[\mathcal{C}]^{\operatorname{op}} \to \operatorname{Set}_{\Delta}^+$, and equip it with the projective model structure. The $(\infty, 1)$ -categorical Grothendieck construction then takes the form of a Quillen equivalence

$$\operatorname{St}_{\mathfrak{C}}: (\operatorname{Set}_{\Delta}^{+})_{/\mathfrak{C}} \iff (\operatorname{Set}_{\Delta}^{+})^{\mathfrak{C}[\mathfrak{C}]^{\operatorname{op}}}: \operatorname{Un}_{\mathfrak{C}}$$

between these two model categories.

In the ∞ -categorical context, Grothendieck constructions have become an indispensable tool, as the added computational complexity of ∞ -categorical constructions renders many ad-hoc constructions of functors nearly impossible to work with. It is often far easier to work with the fibration associated to a functor of ∞ -categories than with the functor itself. Examples of such applications include the study of monoidal $(\infty, 1)$ -categories in [20] and [22] and the approach to lax colimits presented in [14]. The study of higher forms of cofinality, which we will discuss later in this introduction, is another case in which it is virtually essential to use the Grothendieck construction.

The zoo of $(\infty, 1)$ -categorical Grothendieck constructions is complicated by the fact that such Grothendieck constructions come in *two* variances. One can either consider the aforementioned *Cartesian* fibrations of $(\infty, 1)$ -categories over \mathcal{C} , or consider *coCartesian* fibrations over \mathcal{C} . The former correspond to $(\infty, 1)$ -functors

$$F \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{C}\mathrm{at}_{\infty}$$

whereas the latter correspond to $(\infty, 1)$ -functors

$$F: \mathcal{C} \longrightarrow \mathcal{C}at_{\infty}$$

Additionally, if one treats the case of functors

$$F: \mathcal{C}^{\mathrm{op}} \longrightarrow \mathcal{S} \subset \mathcal{C}\mathrm{at}_{\infty}$$

valued in ∞ -groupoids (spaces), one obtains more restrictive variants of Cartesian/coCartesian fibrations, called *right fibrations* and *left fibrations*, respectively, in [21, Ch. 2].

Each variance can be obtained from the other by appropriate dualization proceedures, and so, in practice it is only necessary to prove one correspondence to obtain the other. In the world of $(\infty, 2)$ -categories, where there are *four* possible variances, a similar principle applies, although the dualization procedures can become more complicated. As a result, we have focused on a single variance in our exploration of the $(\infty, 2)$ -categorical Grothendieck construction. Later in the introduction we will give a more complete account of known Grothendieck constructions, as well as a table of relations between them.

1.1.2. The $(\infty, 2)$ -categorical Grothendieck construction

The present paper provides a complete $(\infty, 2)$ -categorical Grothendieck construction. Loosely speaking, for every scaled simplicial set S, we provide an equivalence of $(\infty, 2)$ -bicategories (or simply ∞ -bicategories as in [23])

$$2\mathbb{C}\operatorname{art}(S) \simeq \operatorname{Fun}(S^{\operatorname{op}}, \mathbb{B}\operatorname{icat}_{\infty})$$

between 2-Cartesian fibrations¹ over S, and $(\infty, 2)$ -functors $S^{op} \to \mathbb{B}icat_{\infty}$ with values in ∞ -bicategories.

To understand this construction on an intuitive level, it is helpful to first consider the strict 2-categorical variant of the construction, developed by Buckley in [7]. In this setting, we consider strict 2-functors $p:\mathbb{C}\to\mathbb{D}$. A 1-morphism $f:c\to \overline{c}$ in \mathbb{C} is called *Cartesian* if, for every $a\in\mathbb{C}$ there is a pullback square of categories

$$\mathbb{C}(a,c) \xrightarrow{f_*} \mathbb{C}(a,\overline{c}) \\
P \downarrow \qquad \qquad \downarrow P \\
\mathbb{D}(P(a),P(c)) \xrightarrow{P(f)_*} \mathbb{D}(P(a),P(\overline{c}))$$

A 2-morphism $\alpha: f \Rightarrow g$ in $\mathbb{C}(x,y)$ is called *coCartesian* if it is a coCartesian 1-morphism for the map

$$P \colon \mathbb{C}(x,y) \longrightarrow \mathbb{D}(P(x),P(y)).$$

The functor P is then called a 2-Cartesian fibration if it admits Cartesian lifts of all 1-morphisms, and coCartesian lifts of all 2-morphisms.

In our previous paper, [4], we provided a model structure which we claimed modeled the appropriate $(\infty, 2)$ -categorical variant of the above definitions. To keep track of the data of (1) invertible 2-morphisms, (2) Cartesian 1-morphisms, and (3) coCartesian 2-morphisms in the simplicial setting, we considered a 3-part decoration on simplicial sets. Given a simplicial set $X \in \operatorname{Set}_{\Delta}$, we define a marking and biscaling on X to consist of

- As in [23], invertible 2-morphisms are encoded as a collection $T_X \subset X_2$ of 2-simplices, which is required to contain degenerate simplices. The 2-simplices in T_X are called *thin* 2-simplices.
- As in [21], Cartesian 1-morphisms are encoded as a collection $M_X \subset X_1$ of 1-simplices, which is required to contain the degenerate 1-simplices. The 1-simplices in M_X are called *marked* 1-simplices.
- The coCartesian 2-morphisms are encoded as a collection $C_X \subset X_2$. Since every invertible 2-morphism should be coCartesian, we require that $T_X \subset C_X$. We refer to the 2-simplices in C_X as lean 2-simplices.

A tuple $(X, M_X, T_X \subset C_X)$ is referred to as a marked-biscaled simplicial set (or **MB** simplicial set for short). We denote the category of **MB** simplicial sets by $\operatorname{Set}_{\Delta}^{\mathbf{mb}}$. Summarizing the main results from our paper [4], we have

Theorem. Let (S, T_S) be a scaled simplicial set.

- 1. There is a left proper, combinatorial, simplicial model structure on $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/(S,\sharp,T_S\subset\sharp)}$, called the 2-Cartesian model structure.
- 2. If $S = \Delta^0$ is the terminal scaled simplicial set, the resulting model structure models ∞ -bicategories.
- 3. If (S, T_S) is the scaled nerve of a strict 2-category \mathbb{D} , every 2-Cartesion fibration of strict 2-categories $P: \mathbb{C} \to \mathbb{D}$ gives rise to a fibrant object of $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/(S,\sharp,T_S\subset\sharp)}$.

In the second of these results, our decoration becomes highly redundant. In a fibrant object, the marked 1-morphisms correspond to equivalences, the thin 2-simplices correspond to invertible 2-morphisms, but the lean 2-simplices are identical to the thin 2-simplices. To simplify our later

¹What we call 2-Cartesian fibrations are called *outer 2-Cartesian fibrations* in [12]. Because we focus on a single variance, we trim the terminology for ease of reading.

computations, we rectify this redundancy by also considering marked-scaled simplicial sets, i.e., triples (X, M_X, T_X) consisting of a simplicial set X, a collection of marked 1-simplices M_X , and a collection of thin 2-simplices T_X . The category of marked-scaled simplicial sets is denoted by $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$. The first result of this paper formalizes the fact that marked-scaled simplicial sets should also model ∞ -bicategories.

Theorem. There is a left proper, combinatorial, $\operatorname{Set}_{\Delta}^+$ -enriched model structure on $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$. Moreover, it is Quillen equivalent to the 2-Cartesian model structure on $\operatorname{Set}_{\Delta}^{\mathbf{mb}}$, and thus models ∞ -bicategories.

This can be found in the body of the paper as Theorem 2.45 and Proposition 2.58. The main construction of this paper yields a functor for each scaled simplicial set S

$$\mathbb{S}_{t_S} \colon (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S} \longrightarrow \operatorname{Fun}(\mathfrak{C}^{\operatorname{sc}}[S]^{\operatorname{op}}, \operatorname{Set}_{\Delta}^{\mathbf{ms}})$$

called the bicategorical straightening over S. The functor itself is simply a more highly decorated version of previous straightening functors (e.g., that of [21]), and is discussed in detail at the beginning of section 3. We then show that $\mathbb{S}t_S$ admits a right adjoint $\mathbb{U}n_S$ which we call the (bicategorical) unstraightening over S. As already discussed, the category $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$ carries a model structure which models 2-Cartesian fibrations. If we equip the category of $\operatorname{Set}_{\Delta}^{+}$ -enriched functors $\mathfrak{C}[S]^{\mathrm{op}} \to \operatorname{Set}_{\Delta}^{\mathbf{ms}}$ with the projective model structure, we obtain an enriched model category which models the $(\infty,1)$ -category of $(\infty,2)$ -functors $S^{\mathrm{op}} \to \mathbb{B}\mathrm{icat}_{\infty}$. The main technical result of this paper is that this adjunction is in fact a Quillen equivalence.

Theorem (Theorem 3.85). Let S be an scaled simplicial set. Then the bicategorical straightening-unstraightening adjunction defines a Quillen equivalence

$$\mathbb{S}t_S : (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S} \iff (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}[S]^{\operatorname{op}}} : \mathbb{U}n_S$$

between the model structure on (outer) 2-Cartesian fibrations over S and the projective model structure on $\operatorname{Set}_{\Delta}^+$ -enriched functors $\mathfrak{C}[S]^{\operatorname{op}} \to \operatorname{Set}_{\Delta}^{\mathbf{ms}}$ with values in marked-scaled simplicial sets.

Observe that both model categories are in fact $\operatorname{Set}_{\Delta}^+$ -enriched categories. After performing elementary explicit verifications we prove that the functor \mathbb{U}_{S} is compatible with the (co)tensoring yielding an upgrade of the previous theorem to an intrinsinc bicategorical result.

Theorem (Corollary 3.90). The bicategorical straightening is a left Quillen equivalence for any scaled simplicial set S. Moreover, the functor $\mathbb{U}_{\mathbf{n}_S}$ provides an equivalence of $(\infty, 2)$ -categories

$$2\mathbb{C}\operatorname{art}(S) \simeq \operatorname{Fun}(S^{\operatorname{op}}, \mathbb{B}\operatorname{icat}_{\infty}).$$

The majority of this paper is thus devoted to this proof. We will recapitulate the major ideas of the proof, as well as the structure of the paper, in the penultimate section of the introduction.

1.1.3. A relative 2-nerve

Although it is desirable to have a bicategorical Grothendieck construction that works in the most the general context possible, many practical applications make use of those ∞ -bicategories which arise as scaled nerves of strict 2-categories. We provide a version of the Grothendieck construction better suited to this particular situation in the appendix. In this context, we define an explicit version $\chi_{\mathbb{C}}$ of the unstraightening functor over $N^{\text{sc}}(\mathbb{C})$, which we call the *relative 2-nerve*.

Theorem (Corollary A.20). Let \mathbb{C} be an strict 2-category. Then there is a Quillen equivalence

$$\phi_{\mathbb{C}}: (\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/\operatorname{N}^{\mathrm{sc}}(\mathbb{C})} \, \Longleftrightarrow \, (\mathrm{Set}_{\Delta}^{\mathbf{ms}})^{\mathbb{C}^{(\mathrm{op}, -)}}: \chi_{\mathbb{C}}$$

and an equivalence of left-derived functors² LSt_C $\stackrel{\simeq}{\Longrightarrow}$ L $\phi_{\mathbb{C}}$.

As in the $(\infty, 1)$ -categorical setting (see Section 3.2.5 in [21]) the benefits of a relative nerve construction are twofold: on the one hand, the relative 2-nerve is particularly computationally tractable and well-suited to explicit examples; on the other, the relative 2-nerve allows us to compare our ∞ -bicategorical Grothendieck construction to preexisting strict Grothendieck constructions. We apply our relative nerve construction to obtain a comparison with the Grothendieck construction appearing in [7]. The strict 2-categorical Grothendieck construction of [7] takes the form of an equivalence

$$\mathbb{E}1: \operatorname{Fun}^{\operatorname{ps}}(\mathbb{C}^{\operatorname{op}}, \operatorname{Cat}_2) \longrightarrow 2 \operatorname{Cart}$$

for a 2-category \mathbb{C} . The final result of our appendix shows that relative 2-Nerve coincides with \mathbb{E} l for every strict 2-functor with values in 2-categories.

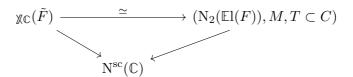
Theorem (Theorem A.21). Let

$$F: \mathbb{C}^{(\mathrm{op},-)} \longrightarrow 2\mathrm{Cat}$$

be a 2-functor, and let \tilde{F} denote the composite

$$\mathbb{C}^{(\mathrm{op},-)} \longrightarrow 2\mathrm{Cat} \longrightarrow \mathrm{Set}_{\Lambda}^{\mathbf{ms}}$$

Then there is an equivalence



of 2-Cartesian fibrations over $N^{sc}(\mathbb{C})$.

1.1.4. The zoo of Grothendieck constructions

To aid the reader in connecting our work here to other results in the literature, we here provide a brief overview of existing Grothendieck constructions, as well as known connections among them. Where practicable, we will choose the version of the construction with the correct variance to agree with our construction.

• The classical Grothendieck construction of [17] takes the form of a equivalence

El:
$$\operatorname{Fun}^{\operatorname{ps}}(C^{\operatorname{op}},\operatorname{Cat}) \longrightarrow \operatorname{Cart}(C)$$

for a 1-category C

• The classical Grothendieck construction is often restricted to categories fibred in groupoids, in which case it takes the form of an equivalence

el:
$$\operatorname{Fun}^{\operatorname{ps}}(C^{\operatorname{op}},\operatorname{Grpd}) \longrightarrow \operatorname{Rfib}(C)$$

for a 1-category C.

• The strict 2-categorical Grothendieck construction of [7] takes the form of an equivalence

$$\mathbb{E}1: \operatorname{Fun}^{\operatorname{ps}}(\mathbb{C}^{\operatorname{op}}, \operatorname{Cat}_2) \longrightarrow 2 \operatorname{Cart}$$

for a 2-category \mathbb{C} .

²see the appendix for a precise definition of $St_{\mathbb{C}}$

- The three Grothendieck-Lurie constructions:
 - For $(\infty, 1)$ -categories fibred in ∞ -groupoids, the construction of [21, Ch 2] takes the form of a left Quillen equivalence

$$\operatorname{St}_S \colon (\operatorname{Set}_{\Delta})_{/S} \longrightarrow (\operatorname{Set}_{\Delta})^{\mathfrak{C}[S]^{\operatorname{op}}}$$

for S a simplicial set. Here $(\operatorname{Set}_{\Delta})_{/S}$ is equipped with the model structure for right fibrations, and $(\operatorname{Set}_{\Delta})^{\mathfrak{C}[S]^{\operatorname{op}}}$ is equipped with the projective model structure obtained from the Kan-Quillen model structure.

- For $(\infty, 1)$ -categories fibred in $(\infty, 1)$ -categories, the construction of [21, Ch. 3] takes the form of a left Quillen equivalence

$$\operatorname{St}_S^+ \colon (\operatorname{Set}_{\Delta}^+)_{/S} \longrightarrow (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}[S]^{\operatorname{op}}}$$

where S is a simplicial set. Here $(\operatorname{Set}_{\Delta}^+)_{/S}$ carries the Cartesian model structure and $(\operatorname{Set}_{\Delta}^+)^{\mathfrak{C}[S]^{\operatorname{op}}}$ carries the projective model structure on $\operatorname{Set}_{\Delta}$ -enriched functors.

– For ∞ -bicategories fibred in $(\infty, 1)$ -categories, the construction of [23] takes the form of a left Quillen equivalence

$$\operatorname{St}_{S}^{(2,1)} : (\operatorname{Set}_{\Delta}^{+})_{/S} \longrightarrow (\operatorname{Set}_{\Delta}^{+})^{\mathfrak{C}^{\operatorname{sc}}[S]^{(\operatorname{op},\operatorname{op})}}$$

where S is a scaled simplicial set. Here $(\operatorname{Set}_{\Delta}^+)_{/S}$ carries the \mathfrak{P} -anodyne model structure of [23, Section 3.2] and $(\operatorname{Set}_{\Delta}^+)^{\mathfrak{C}^{\operatorname{sc}}[S]^{(\operatorname{op},\operatorname{op})}}$ carries the projective model structure on $\operatorname{Set}_{\Delta}^+$ -enriched functors.

• A general (∞, n) -categorical Grothendieck construction is given by Rasekh in [27, Section 5], using Θ_n -spaces. This Grothendieck construction takes the form of a zig-zag of Quillen equivalences, which, given an $(\infty, n+1)$ -category \mathcal{C} presented as a complete Segal object in Θ_n -spaces, induces an equivalence

$$E_{\Theta} \colon \operatorname{Cart}_{(\infty,n)}(\mathfrak{C}) \xrightarrow{\sim} \operatorname{Fun}_{(\infty,n+1)}(\mathfrak{C}, \operatorname{Cat}_{(\infty,n)})$$

between (∞, n) -Cartesian fibrations over \mathcal{C} , and $(\infty, n+1)$ -functors $\mathcal{C} \to \operatorname{Cat}_{(\infty, n)}$.

• The comprehension construction, defined by Riehl and Verity in [28] works in an ∞ -cosmos \mathcal{K} . In the ∞ -cosmos of quasi-categories, it provides a functor

Comp: Fun
$$(B, qCat) \longrightarrow coCart(qCat)_{/B}$$

from functors into small quasi-categories to coCartesians fibrations over B. However, the authors defer the proof that this is an equivalence to a latter work.

We summarize the known relations relations between these constructions in the following diagram. An arrow in the diagram represents a special case, e.g. $El \to St_S^+$ means that El is known to be equivalent to a special case of St_S^+ . A dashed arrow will represent a relation which we conjecture to hold.

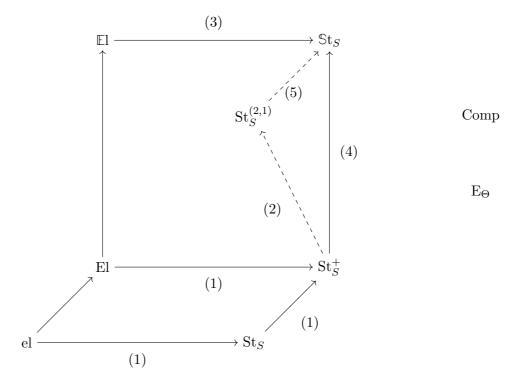
In particular:

- 1. These relations are proven in [21]. In particular, the comparison of the $(\infty, 1)$ -categorical and strict cases passes through the *relative nerve* of section 3.2.5.
- 2. In [23, Remark 4.5.10] the author claims that due to the formal differences in the construction of both straightening functors, no direct comparison seems possible. Instead, the author proves

a comparison on fibrant object without showing naturality in [23, Prop. 4.5.10] using model-independent arguments.

However, we attribute the difficulty of a comparison to the fact that both straightening functors have different variances. It follows from our construction (see point 4 on this list) of St_S that St_S^+ has "morally" an outer Cartesian variance. Since St_S^+ models the Grothendieck construction for ∞ -categories this construction is blind to the variance in 2-morphisms and it is seen as having simply a Cartesian variance. The construction of [23, Prop. 4.5.9] passes through two 1-morphism dualizations to obtain a Cartesian variant of $\operatorname{St}_S^{(2,1)}$. We thus believe a more complete comparison result with the $\operatorname{St}_S^{(2,1)}$ after taking the pertinent 2-morphism dual, as well.

- 3. We show this relation in the Theorem A.21, making use of a relative 2-nerve which we construct for that purpose in the appendix.
- 4. This relation is nearly immediate from the definitions. We give a proof in Proposition 3.15.
- 5. We believe that this relation will hold, however, the constructions $\operatorname{St}_{S}^{(2,1)}$ and St_{S} are related by a 2-morphism dualization. Given the difficulties inherent in realizing 2-morphism duals in scaled simplicial sets, we defer any attempt to prove this statement for the time being.



1.2. Cofinality

The original motivation for this work was to understand the conditions that characterize cofinal functors of ∞ -bicategories. Recall that a functor $f: \mathcal{C} \to \mathcal{D}$ of $(\infty, 1)$ -categories is said to be *cofinal* if, for any ∞ -category \mathcal{E} and any diagram $g: \mathcal{D} \to \mathcal{E}$ the canonical comparison map (whenever defined)

$$\operatorname{colim}_{\mathfrak{C}} g \circ f \xrightarrow{\simeq} \operatorname{colim}_{\mathfrak{D}} g.$$

is an equivalence in \mathcal{E} . In other words, restriction along f preserves all colimits of shape \mathcal{D} .

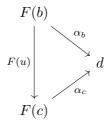
A celebrated theorem of Joyal (in the unpublished work [18]) gives a necessary and sufficient criterion for the cofinality a functor $f: \mathcal{C} \to \mathcal{D}$ as above: f is cofinal if and only if, for every $d \in \mathcal{D}$, the slice $\mathcal{C}_{d/}$ is weakly contractible. Our primary application of the ∞ -bicategorical Grothendieck

construction described above will be a generalization of this result to the $(\infty, 2)$ -categorical setting. The proof of the $(\infty, 1)$ -categorical statement (see Theorem 4.1.3.1 in [21]) uses as a key ingredient the theory of Cartesian fibrations. It was thus expected that a proof of a 2-categorical cofinality theorem should involve higher notions of fibrations as well as the corresponding Grothendieck constructions.

1.2.1. Marked colimits

As one tries to generalize even strict colimits to 2-categories one runs into an immediate problem: which definition of colimit to use. Loosely speaking, any definition of a colimit should come equipped with a universal cone. However, if we consider a strict 2-functor $F: \mathbb{C} \to \mathbb{D}$, we run into an issue defining cones over F. A cone over F with tip d should consist of:

- For every object $c \in \mathbb{C}$, a morphism $\alpha_c : F(c) \to d$.
- For every morphism $u:b\to c$ in \mathbb{C} , a diagram



that commutes appropriately.

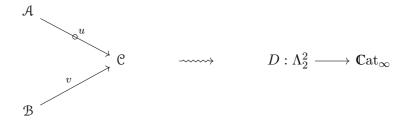
It is here that the definition flounders — there are multiple 2-categorical notions which could be described as the diagram 'commuting appropriately', and each yields different notions of colimit. If one requires the triangles to commute up to non-invertible 2-morphism, for instance, one obtains the notion of a *lax* colimit. If, on the other had, one requires commutativity up to invertible 2-morphism, the corresponding notion of colimit is the *pseudo-colimit*.

One traditional way of resolving this multiplicity of definitions is by defining the more general notion of weighted colimits, which specialize to each of the above cases (see for example [19]). However, in the past two years, a different approach has become relevant due to its amenability to applications in simplicial models for higher categories: marked colimits. In defining marked colimits, one considers the 2-category $\mathbb C$ to be equipped with a collection of marked 1-morphisms, and then requires that the chosen 2-morphism making the triangle above commute is invertible whenever u is a marked morphism. This resolution of the above issue loses nothing in comparison to Cat-weighted or \mathbb{C} at_weighted limits, as the two theories turn out to be equivalent (see [1, Theorem 4.7] and [11, Section 5] for more details). Although this definition of 2-categorical limit is in fact a novel concept in the study of ∞ -category theory its use in the strict 2-categorical realm was already established as in seen in [8].

The reason why this approach is preferred when working with scaled simplicial sets is a familiar one: The marked colimits approach does not require an extra functor (the weight) in its definition and absorbs more information in the diagram. Since elementary constructions with functors of scaled simplicial sets tend to be complicated, this approach yields a more computationally tractable model. This is corroborated by the familiar expression that our 2-categorical conditions for cofinality take.

The degree to which marked colimits are well-suited to higher-categorical settings is underlined by the fact that, within the span of a year, three groups independently arrived at more or less the same definition: the authors of the current paper in [2] and the first author further in [1]; Berman in [6]; and Gagna, Harpaz, and Lanari in [11]. The last of these three provides a complete definition of marked limits and colimits in terms of marked-scaled simplicial sets. Their definitions coincide with those given in [2], [1], and [6] whenever both versions apply.

Before diving into the theory of cofinality let us study some examples of colimits in $\mathbb{C}at_{\infty}$: The ∞ -bicategory of ∞ -categories. Let us consider a diagram



where the circled arrow indicates that the edge $1 \to 2$ in Λ_2^2 is marked. By [1, Theorem 8] we can use the Grothendieck construction to compute the marked colimit of D via a suitable localization. Let $\mathcal{D} = \mathrm{Un}_{\Lambda_2^2}^{\mathrm{co}}(D)$ denote the coCartesian version of the unstraightening functor given in [21]. Informally we can describe the ∞ -category \mathcal{D} as follows:

- The objects of \mathcal{D} are given by pairs of objects (ε, x) where $\varepsilon \in \Lambda_2^2$ and $x \in D(\varepsilon)$.
- A morphism $(\varepsilon_0, x_0) \to (\varepsilon_1, x_1)$ is given by a morphism $\alpha : \varepsilon_0 \to \varepsilon_1$ in Λ_2^2 and a morphism $a : D(\alpha)(x_0) \to x_1$ in $D(\varepsilon_1)$.

We equip \mathcal{D} with a marking by declaring an edge marked if α is the morphism $1 \to 2$ and $a : D(\alpha)(x_0) \to x_1$ is an equivalence in \mathcal{C} . We denote the resulting marked ∞ -category by \mathcal{D}^{\dagger} . The aforementioned theorem then implies that the marked colimit of D is precisely the ∞ -categorical localization $L_W(\mathcal{D}^{\dagger})$.

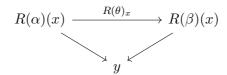
Observe that for every object (1, x) in $L_W(\mathcal{D}^{\dagger})$ is equivalent to an object of the form (2, y). This hints at the possibility that the marked colimit of D can be computed just in terms of the morphism v. In other words, the inclusion $\Delta^{\{0,2\}} \to (\Lambda_2^2)^{\dagger}$ (where the subscript denotes our marking on Λ_2^2) should be marked cofinal. This intuition will be formalized once we present our cofinality theorem.

For our final example we consider a pair of functors $F, G : A \to B$ between strict 1-categories and a natural transformation $\Xi : F \Rightarrow G$ depicted diagramatically as

$$A \xrightarrow{F} B \qquad \qquad R: \mathbb{Q}^2 \longrightarrow \mathbb{C}\mathrm{at} \subset \mathbb{C}\mathrm{at}_{\infty}$$

where \mathbb{Q}^2 is the free-living 2-morphism. We will compute the marked colimit of R where \mathbb{Q}^2 carries the minimal marking, in other words, the lax colimit of R. We define a 2-category \mathbb{R} described as follows:

- Objects are given by pairs (ε, x) where $\varepsilon \in \mathbb{Q}^2$ and $x \in R(\varepsilon)$.
- A morphism from $(\varepsilon_0, x) \to (\varepsilon_1, y)$ is given by a morphism $\alpha : \varepsilon_0 \to \varepsilon_1$ in \mathbb{Q}^2 and a morphism $R(\alpha)(x) \to y$.
- Given a pair of morphisms $u, v : (\varepsilon_0, x) \to (\varepsilon_1, y)$ a 2-morphism $\Theta : u \Rightarrow v$ is given by a 2-morphism $\theta : \alpha \Rightarrow \beta$ in \mathbb{Q}^2 and a commutative diagram in $R(\varepsilon_1)$



Note that Θ is not an identity 2-morphism precisely when $R(\theta) = \Xi$.

The lax colimit of R is then given by the ∞ -category obtained by formally inverting all 2-morphisms in \mathbb{R} . A model for the lax colimit can be obtained as the fibrant replacement of the Duskin nerve (Definition 2.5) of \mathbb{R} in the Joyal's model structure for ∞ -categories.

1.2.2. Marked cofinality

The conditions that characterize cofinal functors of $(\infty,1)$ -categories were first introduced by Quillen in Theorem A, [26]. In this contex, Theorem A gives sufficient conditions for a functor of ordinary 1-categories $f:C\to D$ to induce a homotopy equivalence upon passage to geometric realizations. This theorem has been extensively used in K-Theory and is a key ingredient in the proof of the additivity theorems of Quillen's Q-construction [26] and Waldhausen's S_{\bullet} -construction [30]. The connection between Theorem A and the theory of cofinality can be easily recovered by observing that the geometric realization of a category C can be obtained as the $(\infty,1)$ -colimit of the constant point-valued diagram. With this result established, we see that an $(\infty,1)$ -cofinal functor $f:C\to D$ must necessarily induce an equivalence on geometric realizations. The characterization of $(\infty,1)$ -cofinal functors proven by Joyal then immediately yields Quillen's Theorem A as a corollary.

In [2], we proved a 2-categorical variant of the original Theorem A of Quillen. More precisely, we gave sufficient conditions for a functor of strict 2-categories $f:\mathbb{C}^{\dagger}\to\mathbb{D}^{\dagger}$ to induce a categorical equivalence upon passage to ∞ -categorical localizations. We also provided a treatment of marked colimits in the strict 2-categorical case, and proved a criterion for strict marked 2-cofinality. Furthermore, we conjectured a criterion on slice $(\infty,2)$ -categories which should govern the marked cofinality of $(\infty,2)$ -functors. The conjecture was proven by the first-named author in [1] for the case of marked colimits indexed by an $(\infty,1)$ -category. The second main result of the present paper is a full proof of this cofinality conjecture:

Theorem (Theorem 4.29). Let $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ be a functor of marked ∞ -bicategories. Then the following statements are equivalent

- 1. The functor f is marked cofinal.
- 2. For every $d \in \mathbb{D}$ the functor f induces an equivalence of ∞ -categorical localizations $L_W(\mathbb{C}_{d\uparrow}^{\dagger}) \to L_W(\mathbb{D}_{d\uparrow}^{\dagger})$.
- 3. The following conditions hold:
 - i) For every $d \in \mathbb{D}$ there exists a morphism $g_d : d \to f(c)$ which is initial in $L_W(\mathbb{C}_{d\uparrow}^{\dagger})$ and $L_W(\mathbb{D}_{d\uparrow}^{\dagger})$.
 - ii) Every marked morphism $d \to f(c)$ defines an initial object in $L_W(\mathbb{C}_{d\uparrow}^{\dagger})$.
 - iii) For any marked morphism $d \to b$ in \mathbb{D} the induced functor $L_W(\mathbb{C}_{b\uparrow}^{\dagger}) \to L_W(\mathbb{C}_{d\uparrow}^{\dagger})$ preserves initial objects.

Before commenting on the proof of this theorem, let us observe that in the case where $\mathbb{C} = \mathcal{C}$, $\mathbb{D} = \mathcal{D}$ are ∞ -categories and all morphisms are marked the second statement simply requires that the comma categories $\mathcal{C}_{d/}$ are contractible for every $d \in \mathcal{D}$ thus recovering the usual characterization of cofinal functors of ∞ -categories.

The proof of our cofinality criterion is an adaptation from the proof of Theorem 9 in [1]. First we produce for every morphism of ∞ -bicategories $p: \mathbb{X} \to \mathbb{D}$ a 2-Cartesian fibration $\mathbb{F}(p): \mathbb{F}(\mathbb{X}) \to \mathbb{D}$ whose fibre over an object d can be identified with the 2-categorical variants of the comma category $\mathbb{X}_{d\mathcal{T}}$. We observe that there exists a canonical morphism $\gamma_X: \mathbb{X} \to \mathbb{F}(\mathbb{X})$ which is a 2-Cartesian equivalence over \mathbb{D} . We interpret the map γ_X as the unit of a free-forgetful adjunction and consequently call $\mathbb{F}(p): \mathbb{X} \to \mathbb{D}$ the free 2-Cartesian fibration on p. This allows us to produce a commutative diagram over \mathbb{D} for every morphism of marked ∞ -bicategories $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$

$$\begin{array}{ccc}
\mathbb{F}(\mathbb{C})^{\dagger} & \longrightarrow & \mathbb{F}(\mathbb{D})^{\dagger} \\
& \cong & & \cong & \uparrow \\
\mathbb{C}^{\dagger} & \xrightarrow{f} & \mathbb{D}^{\dagger}
\end{array}$$

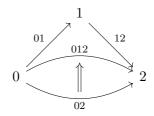
where the vertical morphisms are 2-Cartesian equivalences. The proof follows after carefully translating what it means that f is marked cofinal to the top horizontal morphism and vice versa.

We finish this section by describing simple applications of our cofinality criterion. Firstly, let us return to the first example in the previous section namely the inclusion $\Delta^{\{0,2\}} \to (\Lambda_2^2)^{\dagger}$ where the subscript indicates that the edge $1 \to 2$ is marked. One easily checks that the morphisms

$$\Delta_{i/}^{\{0,2\}} \longrightarrow (\Lambda_2^2)_{i/}^{\dagger}$$

are isomorphisms except in the case where i=1. In this case, the map can be identified with the inclusion of the terminal vertex into $(\Delta^1)^{\sharp}$ and consequently induces an equivalence upon passage to ∞ -categorical localizations.

Lastly, let \mathbb{O}^2 denote the poset-enriched category with objects 0, 1, and 2, morphisms given by chains from i to j in [2], and 2-morphisms given by inclusions. Schematically, this can be drawn as



Let us denote by \mathbb{Q}^2 the full 2-subcategory of \mathbb{O}^2 on the objects 0 and 2. Note that \mathbb{Q}^2 can be described as the free-living 2-morphism considered in the previous section. We define a marking on \mathbb{O}^2 by declaring 12 to be marked, and consider \mathbb{Q}^2 to be minimally marked. This gives us a functor

$$\iota \colon (\mathbb{Q}^2)^{\flat} \longrightarrow (\mathbb{O}^2)^{\dagger}$$

of marked 2-categories. Intuitively, it is clear that a marked cone over a functor out of $(\mathbb{O}^2)^{\dagger}$ should be equivalent to a marked cone over $(\mathbb{Q}^2)^{\flat}$, owing to the necessity that the 2-morphism of the cone corresponding to 12 be invertible. To see this formally, we can use the criterion above by computing the slices of ι .

• The induced map

$$(\mathbb{Q}^2)^{\flat}_{27} \longrightarrow (\mathbb{O}^2)^{\dagger}_{27}$$

is the identity on the terminal 2-category.

• The induced map

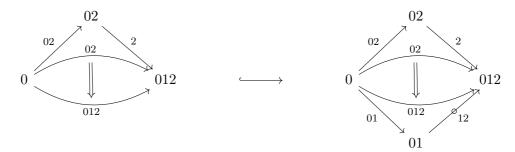
$$(\mathbb{Q}^2)_{1\uparrow}^{\flat} \longrightarrow (\mathbb{O}^2)_{1\uparrow}^{\dagger}$$

is the inclusion of the terminal vertex in $(\Delta^1)^{\sharp}$.

• The induced map

$$(\mathbb{Q}^2)_{0\uparrow}^{\flat} \longrightarrow (\mathbb{O}^2)_{0\uparrow}^{\dagger}$$

Is, pictorially, the inclusion



where the circled arrow denotes a marked morphism. This is a marked homotopy equivalence and thus induces an equivalence on $(\infty, 1)$ -localizations.

Thus, our intuition is confirmed — the 2-functor so defined is indeed marked cofinal.

1.3. A bird's-eye view of the main argument

Having now provided an overview of the main themes and results of the paper, let us attempt to ease the reader's journey through the main arguments by providing a roadmap of sorts.

Leaving aside the question of constructing the model structures, the first step is writing down our putative left Quillen equivalence

$$\operatorname{St}_S \colon (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S} \longrightarrow (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}[S]^{\operatorname{op}}}$$

for a scaled simplicial set X. The construction of this functor amounts to simply providing decorations on the $(\infty, 1)$ -categorical Grothendieck construction St^+ of [21, §3.2], and checking that they are compatible with the desired functoriality, which we do at the beginning of Section 3. The existence of a right adjoint to St_S follows from the adjoint functor theorem. To show that St_S preserves cofibrations amounts to computing its values on a generating set of monomorphisms, and applying base change properties.

The first technical challenge comes in showing that $\mathbb{S}t$ preserves trivial cofibrations. We first show that it sends anodyne morphisms in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$ to trivial cofibrations, again checking on generators and applying base change. With that done, we show that $\mathbb{S}t_S$ weakly preserves the $\operatorname{Set}_{\Delta}^+$ tensoring. Since $\mathbb{S}t_S$ preserves marked anodyne morphisms, a modification of the argument of [21, Cor 3.2.1.16] allows us to reduce the problem of showing that $\mathbb{S}t_S$ is left Quillen to showing that $\mathbb{S}t_S$ sends homotopy equivalences to weak equivalences. However, since $\mathbb{S}t_S$ preserves the tensoring, it sends homotopic maps to homotopic maps, allowing us to conclude that it is left Quillen in Theorem 3.49.

To show that St_S is a Quillen *equivalence*, we perform the kind of dimensional induction used in [21, Section 3.2]:

- We prove that \mathbb{S}_{Δ^0} is a left Quillen equivalence in Section 3.4. The proof of this fact is quite direct, and proceeds by constructing a natural equivalence from \mathbb{S}_{Δ^0} to a more canonical left Quillen equivalence $L \colon \operatorname{Set}_{\Delta}^{\mathbf{mb}} \longrightarrow \operatorname{Set}_{\Delta}^{\mathbf{ms}}$ (which is defined in subsection 2.3).
- We then prove that $\operatorname{St}_{\Delta^n_{\flat}}$ is a left Quillen equivalence in subsection 3.5. This is the most technically demanding step in the proof. We first show that, for any 2-Cartesian fibration $p: X \to \Delta^n_{\flat}$, there is a homotopy pushout diagram

$$X \times (\Delta^{n-1})^{\diamond} \longrightarrow X_n \times (\Delta^n)^{\diamond}$$

$$\downarrow \qquad \qquad \downarrow$$

$$X \times_{\Delta^n} (\Delta^{n-1})^{\diamond} \longrightarrow X$$

in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})/\Delta_{\flat}^{n}$. This statement appears as Corollary 3.80, the proof of which occupies most of Section 3.5.

Once the homotopy pushout is established, we can show that, for a 2-Cartesian fibration $X \to \Delta_{\flat}$ with *i*-fibre X_i , the canonical map $\operatorname{St}_{\Delta^0} \to \operatorname{St}_{\Delta^n_{\flat}}(X)(i)$ is an equivalence. We then use the fibrewise nature of equivalences in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\Delta^n_{\flat}}$ together with the pointwise nature of equivalences in $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[\Delta^n_{\flat}]^{\mathrm{op}}}$ to complete the proof.

• The general case follows from the special cases over simplices by an inductive argument nearly identical to that of [23, Prop. 3.8.4].

The large-scale form of the argument is thus a fairly familiar one — the technical arguments below serve to fill in the scaffold described above.

1.4. Structure of the paper

The paper is structured as follows. In Section 2 we fix notational conventions and recall the model structure on $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$ from [4]. We then define the model structure on $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ and show that it is Quillen equivalent to the model structures on $\operatorname{Set}_{\Delta}^{\mathbf{sc}}$ and $\operatorname{Set}_{\Delta}^{\mathbf{mb}}$.

In Section 3 we define the bicategorical straightening functor, and prove that it is a Quillen equivalence. As described above, we first show that St_S is left Quillen. We show this together with the fact that St_{Δ^0} is a Quillen equivalence at the end of Section 3.4. In Section 3.5, we show that St_S is a Quillen equivalence over an arbitrary simplex. Finally, in Section 3.6, we prove that St_S is a Quillen equivalence in general. This result appears as Theorem 3.85.

We then turn to the proof of the cofinality conjecture in Section 4. In Section 4.1, we construct the free 2-Cartesian fibration on a morphism of ∞ -bicategories $\mathbb{X} \to \mathbb{D}$, and explore some of its properties. In Section 4.2, we prove a sufficient and necessary condition for marked $(\infty, 2)$ -cofinality as Theorem 4.29, using free fibrations to ease our examination of the requisite representability properties.

Finally, in Appendix A, we give a simpler version of our Grothendieck construction over the scaled nerve of a strict 2-category. Using this relative 2-nerve, we show that, in the strict 2-categorical case, our Grothendieck construction restricts to that of [7].

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2. Preliminaries

We here collect background information and preliminary results which will be necessary for our arguments in the rest of the paper. While we will review some 2-category theory, it is impracticable to recapitulate all of the needed background on $(\infty, 2)$ -categories, so we will limit ourselves to a brief discussion of scaled simplicial sets, and direct the reader to [23] for more comprehensive background.

2.1. 2-categories

We will often use strict 2-categories as a tool to explore ∞ -bicategories. In this section, we briefly collect some notations and constructions we will use in the sequel.

Notation. By a 2-category, we will always mean a strict 2-category. By a 2-functor, we will mean a strict 2-functor unless specified otherwise. We will denote strict 2-categories by blackboard bold letters, e.g. \mathbb{D} .

A 2-category \mathbb{D} has three duals, determined by reversing the direction of 1-morphisms, 2-morphisms, or both, respectively. In this work, we will primarily make use of the 1-morphism dual. To better accord with the notation used in (scaled) simplicial sets, we will denote the 1-morphism dual simply by \mathbb{D}^{op} . Where needed, we denote the 2-morphism dual by $\mathbb{D}^{(-,\text{op})}$, and the dual which reverses both 1- and 2-morphisms by $\mathbb{D}^{(\text{op},\text{op})}$. We will denote the 1-category of 2-categories and strict 2-functors by 2Cat.

Definition 2.1. Let I be a linearly ordered finite set. We define a 2-category \mathbb{O}^I as follows

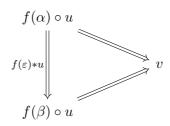
- the objects of \mathbb{O}^I are the elements of I,
- the category $\mathbb{O}^{I}(i,j)$ of morphisms between objects $i,j \in I$ is defined as the poset of finite sets $S \subseteq I$ such that $\min(S) = i$ and $\max(S) = j$ ordered by inclusion,
- the composition functors are given, for $i, j, l \in I$, by

$$\mathbb{O}^{I}(i,j) \times \mathbb{O}^{I}(j,l) \to \mathbb{O}^{I}(i,l), \quad (S,T) \mapsto S \cup T.$$

When I = [n], we denote \mathbb{O}^I by \mathbb{O}^n . Note that the \mathbb{O}^n form a cosimplicial object in 2Cat, which we denote by \mathbb{O}^{\bullet} .

Definition 2.2. Let $f: \mathbb{C} \to \mathbb{D}$ be a functor of 2-categories. Given an object $d \in \mathbb{D}$ we define the lax slice $\mathbb{C}_{d^{\gamma}}$ 2-category as follows:

- Objects are morphisms $u:d\to f(c)$ in $\mathbb D$ with source d such that $c\in\mathbb C.$
- A 1-morphism from $u: d \to f(c)$ to $v: d \to f(c')$ is given by a 1-morphism $\alpha: c \to c'$ in $\mathbb C$ and a 2-morphism $f(\alpha) \circ u \Rightarrow v$.
- A 2-morphism in $\mathbb{C}_{d\uparrow}$ is given by a 2-morphism $\varepsilon:\alpha\Rightarrow\beta$ such that the diagram below commutes



If the functor f is the identity on the 2-category \mathbb{D} we will use the notation $\mathbb{D}_{d^{\uparrow}}$.

Example 2.3. Let I be a linearly ordered finite set and denote its minimum by i. Unraveling Definition 2.2 we see that $\mathbb{O}^I_{i\uparrow}$ is the 2-category given by:

- Objects are subsets $S \subseteq I$ such that $\min(S) = i$.
- A morphism $S \to T$ is given by a subset $U \subseteq I$ such that $\max(S) = \min(U)$ and $\max(U) = \max(T)$ and such that $S \cup U \subseteq T$.
- We have a 2-morphism $U \Rightarrow V$ precisely if $U \subseteq V$.

Remark 2.4. We observe that the non-empty mapping categories in $\mathbb{O}_{i\uparrow}^{I}$ are all contractible since each has an initial object.

To represent 2-categories as simplicial sets, we need a nerve operation. We denote the category of simplicial sets by $\operatorname{Set}_{\Delta}$.

Definition 2.5. Given $\mathbb{D} \in 2Cat$, we define a simplicial set $N_2(\mathbb{D}) \in Set_{\Delta}$, the *Duskin nerve* of \mathbb{D} , by

$$N_2(\mathbb{D})_n := 2Cat(\mathbb{O}^n, \mathbb{D}).$$

2.2. Higher categories and decorated simplicial sets

Throughout this paper, we will make extensive use of models for higher categories in terms of decorated simplicial sets. For models for $(\infty, 1)$ -categories, we will direct the reader to [21, §2.2.5, §3.1], though we briefly discuss notation here.

Definition 2.6. We denote the category of simplicial sets by $\operatorname{Set}_{\Delta}$. A marked simplicial set is defined to be a pair (X, M_X) consisting of a simplicial set $X \in \operatorname{Set}_{\Delta}$, and a collection $M_X \subseteq X_1$ of 1-simplices in X which contains the degenerate 1-simplices. We will denote the category of marked simplicial sets by $\operatorname{Set}_{\Delta}^+$.

We will typically view $\operatorname{Set}_{\Delta}$ as equipped with either the Kan-Quillen model structure (see, e.g., [15, Ch. 1]) or the Joyal model structure (see, e.g. [21, §2.2.5]). We will view $\operatorname{Set}_{\Delta}^+$ as equipped with the Cartesian model structure of [21, §3.1].

The first model structure we will use to study ∞ -bicategories is a model structure on enriched categories.

Notation. We denote by $\operatorname{Cat}_{\Delta}^+$ the category of $\operatorname{Set}_{\Delta}^+$ -enriched categories.

Proposition 2.7. There is a left-proper, combinatorial model structure on $\operatorname{Cat}_{\Delta}^+$ such that

- W The weak equivalences are those enriched functors which are essentially surjective on homotopy categories and induce equivalences on all mapping spaces.
- C The cofibrations the smallest weakly saturated class containing $\varnothing \to [0]_{\operatorname{Set}_{\Delta}^+}$, and each inclusion $[1]_A \to [1]_B$ where $A \to B$ is a generating cofibration for $\operatorname{Set}_{\Delta}^+$.

Proof. This is a special case of [21, A.3.2.4].

Remark 2.8. Notice that, given a strict 2-category \mathbb{D} , we can take the nerves of the mapping categories to obtain a $\operatorname{Set}_{\Delta}$ -enriched category. If we define a marking on each mapping category by declaring precisely the isomorphisms to be marked, we obtain a canonical element of $\operatorname{Cat}_{\Delta}^+$ associated to \mathbb{D} . We will uniformly abuse notation by denoting this $\operatorname{Set}_{\Delta}^+$ -enriched category by \mathbb{D} as well.

The basic idea for the main models for ∞ -bicategories used in this paper is that simplicial sets may be used to model ∞ -bicategories, provided we keep track of which 2-simplices are considered to represent invertible 2-morphisms.

Definition 2.9. A scaled simplicial set consists of a pair (X, T_X) , where $X \in \operatorname{Set}_{\Delta}$ is a simplicial set, and $T_X \subseteq X_2$ is a collection of 2-simplices — called the *thin* 2-simplices — which contains all degenerate 2-simplices. We denote by $\operatorname{Set}_{\Delta}^{\mathbf{sc}}$ the category of scaled simplicial sets.

Notation. We will sometimes make use of subscripts to denote scalings. In particular, (X, X_2) will be will be denoted by X_{\sharp} , and $(X, \deg(X_2))$ will be denoted by X_{\flat} . Similarly, we will sometimes use superscripts to denote markings on a simplicial set: $(X, X_1) = X^{\sharp}$, and $(X, \deg(X_1)) = X^{\flat}$.

Definition 2.10. The set of generating scaled anodyne maps S is the set of maps of scaled simplicial sets consisting of:

(i) the inner horns inclusions

$$(\Lambda_i^n, \{\Delta^{\{i-1,i,i+1\}}\}) \to (\Delta^n, \{\Delta^{\{i-1,i,i+1\}}\})$$
, $n \ge 2$, $0 < i < n$;

(ii) the map

$$(\Delta^4, T) \to (\Delta^4, T \cup \{\Delta^{\{0,3,4\}}, \Delta^{\{0,1,4\}}\}),$$

where we define

$$T \stackrel{\text{def}}{=} \{\Delta^{\{0,2,4\}}, \ \Delta^{\{1,2,3\}}, \ \Delta^{\{0,1,3\}}, \ \Delta^{\{1,3,4\}}, \ \Delta^{\{0,1,2\}}\};$$

(iii) the set of maps

$$\left(\Lambda_0^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\right) \to \left(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \{\Delta^{\{0,1,n\}}\}\right) \quad , \quad n \geqslant 3.$$

A general map of scaled simplicial set is said to be *scaled anodyne* if it belongs to the weakly saturated closure of S.

Definition 2.11. We say that a map of scaled simplicial sets $p: X \to S$ is a weak **S**-fibration if it has the right lifting property with respect to the class of scaled anodyne maps. We call $X \in \operatorname{Set}_{\Delta}^{\mathbf{sc}}$ a ∞ -bicategory if the unique map $X \to \Delta^0_{\sharp}$ is a weak **S**-fibration.

Definition 2.12. The composite

$$\Delta \xrightarrow{\quad \mathfrak{C} \quad} \operatorname{Cat}_{\Delta} \xrightarrow{\quad \flat \quad} \operatorname{Cat}_{\Delta}^+$$

gives us a cosimplicial object in $\operatorname{Cat}_{\Delta}^+$. We can moreover send the thin 2-simplex Δ_{\sharp}^2 to $\mathfrak{C}[\Delta^2]$ equipped with sharp-marked mapping spaces. The usual machinery of nerve and realization then gives us adjoint functors

which we will call the scaled nerve and scaled rigidification.

Remark 2.13. Given a 2-category \mathbb{D} , we can define a scaling on $N_2(\mathbb{D})$ by declaring a triangle to be thin if and only if the corresponding 2-morphism is invertible. Viewing \mathbb{D} as an $\operatorname{Set}_{\Delta}^+$ -enriched category as in Remark 2.8, we see that this scaled simplicial set coincides with $\operatorname{N}^{\operatorname{sc}}(\mathbb{D})$. We thus are justified in speaking of the *scaled nerve* of a 2-category.

Theorem 2.14. There is a left proper, combinatorial model structure on Set_{∞}^{Sc} with

- W The weak equivalences are the morphisms $f: A \to B$ such that $\mathfrak{C}^{\mathrm{sc}}[f]: \mathfrak{C}^{\mathrm{sc}}[A] \to \mathfrak{C}^{\mathrm{sc}}[B]$ is an equivalence in Cat_{Δ}^+ .
- C The cofibrations are the monomorphisms.

Moreover, the fibrant objects in this model structure are the ∞ -bicategories, and the adjunction

is a Quillen equivalence.

Proof. This is [23, Thm A.3.2.4]. The characterization of fibrant objects is [10, Thm 5.1]. \Box

Remark 2.15 (Key notational convention). When no confusion is likely to arise, we denote decorated simplicial sets by simple roman majescules: X, Y, Z, etc. For fibrant objects, representing (higher) categories of various kinds, we fix the following conventions:

- Strict 1-categories will be denoted by undecorated roman majescules, B, C, D, etc.
- $(\infty, 1)$ -categories as presented by Joyal fibrant simplicial sets of fibrant marked simplicial sets will be denoted by calligraphic majescules: \mathcal{B} , \mathcal{C} , \mathcal{D} , etc.
- Strict 2-categories will be denoted by blackboard-bold majescules, B, C, D, etc.
- $(\infty, 2)$ -categories, presented as fibrant scaled simplicial sets, whill be denoted by thickened blackboard-bold majescules: \mathbb{B} , \mathbb{C} , \mathbb{D} , etc.

2.3. Marked biscaled simplicial sets and 2-Cartesian fibrations.

In this section we collect useful definitions and results introduced in [4] that will play a relevant role in this paper.

Definition 2.16. A marked biscaled simplicial set (mb simplicial set) is given by the following data

- A simplicial set X.
- A collection of edges $E_X \in X_1$ containing all degenerate edges.
- A collection of triangles $T_X \in X_2$ containing all degenerate triangles. We will refer to the elements of this collection as thin triangles.
- A collection of triangles $C_X \in X_2$ such that $T_X \subseteq C_X$. We will refer to the elements of this collection as lean triangles.

We will denote such objects as triples $(X, E_X, T_X \subseteq C_X)$. A map $(X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ is given by a map of simplicial sets $f: X \to Y$ compatible with the collections of edges and triangles above. We denote by $\text{Set}^{\mathbf{mb}}_{\Lambda}$ the category of mb simplicial sets.

Notation. Let $(X, E_X, T_X \subseteq C_X)$ be a mb simplicial set. Suppose that the collection E_X consist only of degenerate edges. Then we fix the notation $(X, E_X, T_X \subseteq C_X) = (X, \flat, T_X \subseteq E_X)$ and do similarly for the collection T_X . If C_X consists only of degenerate triangles we fix the notation $(X, E_X, T_X \subseteq C_X) = (X, E_X, \flat)$. In an analogous fashion we will use the symbol " \sharp " to denote a collection containing all edges (resp. all triangles). Finally suppose that $T_X = C_X$ then we will employ the notation (X, E_X, T_X) .

Remark 2.17. We will often abuse notation when defining the collections E_X (resp. T_X , resp. C_X) and just specified its non-degenerate edges (resp. triangles).

Definition 2.18. The set of *generating mb anodyne maps* **MB** is the set of maps of mb simplicial sets consisting of:

(A1) The inner horn inclusions

$$\left(\Lambda^n_i, \flat, \{\Delta^{\{i-1,i,i+1\}}\}\right) \to \left(\Delta^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}\right) \quad, \quad n \geqslant 2 \quad, \quad 0 < i < n;$$

(A2) The map

$$(\Delta^4, \flat, T) \to (\Delta^4, \flat, T \cup \{\Delta^{\{0,3,4\}}, \Delta^{\{0,1,4\}}\}),$$

where we define

$$T \stackrel{\text{def}}{=} \{ \Delta^{\{0,2,4\}}, \ \Delta^{\{1,2,3\}}, \ \Delta^{\{0,1,3\}}, \ \Delta^{\{1,3,4\}}, \ \Delta^{\{0,1,2\}} \};$$

(A3) The set of maps

$$\left(\Lambda^n_0 \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \{\Delta^{\{0,1,n\}}\}\right) \to \left(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \flat \subset \{\Delta^{\{0,1,n\}}\}\right) \quad , \quad n \geqslant 2.$$

These maps force left-degenerate lean-scaled triangles to represent coCartesian edges of the mapping category.

(A4) The set of maps

$$\left(\Lambda_n^n, \{\Delta^{\{n-1,n\}}\}, \flat \subset \{\Delta^{\{0,n-1,n\}}\}\right) \to \left(\Delta^n, \{\Delta^{\{n-1,n\}}\}, \flat \subset \{\Delta^{\{0,n-1,n\}}\}\right) \quad , \quad n \geqslant 2.$$

This forces the marked morphisms to be p-Cartesian with respect to the given thin and lean triangles.

(A5) The inclusion of the terminal vertex

$$\left(\Delta^0,\sharp,\sharp\right) \to \left(\Delta^1,\sharp,\sharp\right).$$

This requires p-Cartesian lifts of morphisms in the base to exist.

(S1) The map

$$\left(\Delta^2, \{\Delta^{\{0,1\}}, \Delta^{\{1,2\}}\}, \sharp\right) \to \left(\Delta^2, \sharp, \sharp\right),$$

requiring that p-Cartesian morphisms compose across thin triangles.

(S2) The map

$$\left(\Delta^2, \flat, \flat \subset \sharp\right) o \left(\Delta^2, \flat, \sharp\right),$$

which requires that lean triangles over thin triangles are, themselves, thin.

(S3) The map

$$\left(\Delta^3, \flat, \{\Delta^{\{i-1,i,i+1\}}\} \subset U_i\right) \to \left(\Delta^3, \flat, \{\Delta^{\{i-1,i,i+1\}}\} \subset \sharp\right) \quad , \quad 0 < i < 3$$

where U_i is the collection of all triangles except *i*-th face. This and the next two generators serve to establish composability and limited 2-out-of-3 properties for lean triangles.

(S4) The map

$$\left(\Delta^{3} \coprod_{\Delta^{\{0,1\}}} \Delta^{0}, \flat, \flat \subset U_{0}\right) \to \left(\Delta^{3} \coprod_{\Delta^{\{0,1\}}} \Delta^{0}, \flat, \flat \subset \sharp\right)$$

where U_0 is the collection of all triangles except the 0-th face.

(S5) The map

$$\left(\Delta^3, \{\Delta^{\{2,3\}}\}, \flat \subset U_3\right) \to \left(\Delta^3, \{\Delta^{\{2,3\}}\}, \flat \subset \sharp\right)$$

where U_3 is the collections of all triangles except the 3-rd face.

(E) For every Kan complex K, the map

$$(K, \flat, \sharp) \to (K, \sharp, \sharp).$$

Which requires that every equivalence is a marked morphism.

A map of mb simplicial sets is said to be MB-anodyne if it belongs to the weakly saturated closure of MB.

Definition 2.19. Let $f:(X, E_X, T_X \subseteq C_X) \to (Y, E_Y, T_Y \subseteq C_Y)$ be a map of mb simplicial sets. We say that f is a **MB**-fibration if it has the right lifting property against the class of **MB**-anodyne morphisms.

Definition 2.20. Given two mb simplicial sets $(K, E_K, T_K \subseteq C_K), (X, E_X, T_X \subseteq C_X)$ we define another mb simplicial set denoted by Fun^{mb}(K, X) and characterized by the following universal property

$$\operatorname{Hom}_{\operatorname{Set}_{\Lambda}^{\mathbf{mb}}} \left(A, \operatorname{Fun}^{\mathbf{mb}}(K, X) \right) \simeq \operatorname{Hom}_{\operatorname{Set}_{\Lambda}^{\mathbf{mb}}} \left(A \times K, X \right).$$

Proposition 2.21. Let $f:(X,E_X,T_X\subseteq C_X)\to (Y,E_Y,T_Y\subseteq C_Y)$ be a MB-fibration. Then for every $K\in \operatorname{Set}_{\Delta}^{\mathbf{mb}}$ the induced morphism $\operatorname{Fun}^{\mathbf{mb}}(K,X)\to \operatorname{Fun}^{\mathbf{mb}}(K,Y)$ is a MB-fibration.

Definition 2.22. Let $f: Y \to S$ be a morphism of mb simplicial another map $g: X \to Y$. We define a mb simplicial set $\operatorname{Map}_Y(K, X)$ by means of the pullback square

$$\begin{array}{ccc} \operatorname{Map}_Y(K,X) & \longrightarrow & \operatorname{Fun}^{\mathbf{mb}}(K,X) \\ \downarrow & & \downarrow \\ \Delta^0 & \xrightarrow{g} & \operatorname{Fun}^{\mathbf{mb}}(K,Y) \end{array}$$

If $f: X \to Y$ is a **MB**-fibration then it follows from the previous proposition that $\operatorname{Map}_Y(K, X)$ is an ∞ -bicategory.

Let $S \in \operatorname{Set}_{\Delta}^{\mathbf{sc}}$ for the rest of the section we will denote $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$ the category of mb simplicial set over $(S, \sharp, T_S \subset \sharp)$.

Definition 2.23. We say that an object $\pi: X \to S$ in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$ is an *outer 2-Cartesian* fibration if it is a **MB**-fibration.

Remark 2.24. We will frequently abuse notation and refer to outer 2-Cartesian as 2-Cartesian fibrations.

Definition 2.25. Let $\pi: X \to S$ be a morphism of mb simplicial sets. Given an object $K \to S$, we define $\operatorname{Map}_{S}^{\operatorname{th}}(K,X)$ to be the mb sub-simplicial set consisting only of the thin triangles. Note that if π is a 2-Cartesian fibration this is precisely the underlying ∞ -category of $\operatorname{Map}_{S}(K,X)$.

We similarly denote by $\operatorname{Map}_{\widetilde{S}}^{\simeq}(K,X)$ the mb sub-simplicial set consisting of thin triangles and marked edges. As before, we note that if π is a 2-Cartesian fibration, the simplicial set $\operatorname{Map}_{\widetilde{S}}^{\simeq}(K,X)$ can be identified with the maximal Kan complex in $\operatorname{Map}_{S}(K,X)$.

Definition 2.26. We define a functor $I: \operatorname{Set}_{\Delta}^{+} \longrightarrow \operatorname{Set}_{\Delta}^{\mathbf{mb}}$ mapping a marked simplicial set (K, E_K) to the mb simplicial set (K, E_K, \sharp) . If K is maximally marked we adopt the notation $I^{+}(K^{\sharp}) = K_{\sharp}^{\sharp}$

Remark 2.27. Note that we can endow the $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$ with the structure of a $\operatorname{Set}_{\Delta}^{+}$ -enriched category by means of $\operatorname{Map}_{S}^{\mathrm{th}}(-,-)$. In addition given $K \in \operatorname{Set}_{\Delta}^{+}$ and $\pi: X \to S$ we define $K \otimes X := I(K) \times X$ equipped with a map to S given by first projecting to X and then composing with π . This construction shows that $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$ is tensored over $\operatorname{Set}_{\Delta}^{+}$. One can easily show that $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$

is also cotensored over $\operatorname{Set}_{\Lambda}^+$.

In a similar way one can use $\operatorname{Map}_{\overline{S}}^{\sim}(-,-)$ to endow $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$ with the structure of a $\operatorname{Set}_{\Delta}$ -enriched category. In this case the cotensor is given by $K \otimes X = I(K^{\sharp}) \times X$.

Definition 2.28. Let $L \xrightarrow{h} K \xrightarrow{p} S$ be a morphism in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$. We say that h is a cofibration when it is a monomorphism of simplicial sets. We will call h a weak equivalence if for every 2-Cartesian fibration $\pi: X \to S$ the induced morphism

$$h^*: \operatorname{Map}_S(K, X) \longrightarrow \operatorname{Map}_S(L, X)$$

is a bicategorical equivalence.

For the convenience of the reader, we here recall the main result of [4]:

Theorem 2.29 ([4] **Theorem 3.38).** Let S be a scaled simplicial set. Then there exists a left proper combinatorial simplicial model structure on $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$, which is characterized uniquely by the following properties:

- C) A morphism $f: X \to Y$ in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$ is a cofibration if and only if f induces a monomorphism betwee the underlying simplicial sets.
- F) An object $X \in (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$ is fibrant if and only if X is a 2-Cartesian fibration.

2.3.1. MB-anodyne morphisms and dull subsets

Before proceeding, we here record two variants of the pivot point trick [5, Lem. 1.10] which will be of use later.

Definition 2.30. Let $\mathbb{P}(n)$ be the power set of [n]. Given $\mathcal{A} \subset \mathbb{P}(n)$ and $X \in \mathbb{P}(n)$ we say that $X \subset \mathcal{A}$ is \mathcal{A} -basal if it contains precisely one element from each $S \in \mathcal{A}$. We denote the set of \mathcal{A} -basal sets by $\text{Bas}(\mathcal{A})$.

Definition 2.31. Given subset $A \subset \mathbb{P}(n)$ such that $\emptyset \notin A$, and a marked-biscaled simplex $(\Delta^n)^{\dagger}$, we define a marked-biscaled simplicial subset

$$(\mathcal{S}^{\mathcal{A}})^{\dagger} = \bigcup_{S \in \mathcal{A}} \Delta^{[n] \setminus S}.$$

Definition 2.32. We call a subset $A \subset \mathbb{P}(n)$ inner-dull if the following conditions are satisfied

- 1. \mathcal{A} does not contain \varnothing .
- 2. There exists 0 < i < n such that $i \notin S$ for every $S \in A$.
- 3. For any $S, T \in \mathcal{A}, S \cap T = \emptyset$.
- 4. For every A-basal set $X \in \mathbb{P}(n)$ there exists $u, v \in X$ such that u < i < v.

We call the element i in the second condition the pivot point.

Definition 2.33. Given an inner-dull subset $A \subset \mathbb{P}(n)$, we define \mathfrak{M}_A to be the set of subsets $X \in \mathbb{P}(n)$ satisfying:

- A1) X contains the pivot point $i \in X$.
- A2) The simplex $\sigma_X : \Delta^X \to (\Delta^n)^{\dagger}$ does not factor through $(\mathcal{S}^{\mathcal{A}})^{\dagger}$.

We define $\mathcal{M}_{\mathcal{A}}^{j} = \{X \in \mathcal{M}_{\mathcal{A}} \mid |X| = j\}$. Note that those elements $X \in \mathcal{M}_{\mathcal{A}}$ of minimal cardinality are of the form $X_0 \cup \{i\}$ for $X_0 \in \text{Bas}(\mathcal{A})$.

Definition 2.34. Let $\mathcal{A} \subset \mathbb{P}(n)$ be an inner-dull subset with pivot point i. Given an \mathcal{A} -basal subset X we denote by $l^X < u^X$ the pair of consecutive elements such that $l^X < i < u^X$.

Lemma 2.35 (The pivot trick). Let $A \subset \mathbb{P}(n)$ be an inner-dull subset and let $(\Delta^n)^{\dagger}$ be a marked biscaled simplex. Suppose that the following conditions hold:

- 1. Every marked edge (resp. thin triangle) which does not contain the pivot point i factors through $(S^A)^{\dagger}$.
- 2. For every $X \in \text{Bas}(A)$ and every $l^X \leqslant r < i < s \leqslant u^X$ the triangle $\{r, i, s\}$ is thin.
- 3. Let $\sigma = \{a < b < c\}$ be a lean simplex not containing the pivot point i. Then either σ factors through $(\mathcal{S}^{\mathcal{A}})^{\dagger}$ or we have a < i < c and the simplex $\sigma \cup \{i\}$ is fully lean scaled.

Then the inclusion

$$(\mathcal{S}^{\mathcal{A}})^{\dagger} \longrightarrow (\Delta^n)^{\dagger}$$

is in the weakly saturated hull of morphisms of type (A1) and (S3).

Proof. Observe that since \mathcal{A} is inner-dull it follows that every \mathcal{A} -basal set has the same cardinality which we denote ε . For every $\varepsilon \leqslant j \leqslant n$ we define

$$Y_j = Y_{j-1} \cup \bigcup_{X \in \mathcal{M}_{\mathcal{A}}^j} \sigma_X$$

where $Y_{\varepsilon-1} = (\mathcal{S}^{\mathcal{A}})^{\dagger}$ and we view σ_X as having the inherited decorations. This yields a filtration

$$(\mathcal{S}^{\mathcal{A}})^{\dagger} \to Y_{\varepsilon} \to \cdots \to Y_{n-1} \to (\Lambda_{i}^{n})^{\dagger} \to (\Delta^{n})^{\dagger}$$

We will show that each step of this filtration can be obtained as an iterated pushout along morphisms of type (A1). Let $X \in \mathcal{M}_A^j$ for $\varepsilon \leqslant j \leqslant n-1$ and consider the pullback diagram

$$\begin{array}{ccc}
\Lambda_i^X & \longrightarrow \Delta^X \\
\downarrow & & \downarrow \sigma_X \\
Y_{j-1} & \longrightarrow Y_j
\end{array}$$

We claim that the top horizontal morphism is in the weakly saturated hull of morphisms of type (A1) and (S3). First we notice that the triangle $\{i-1,i,i+1\}$ is thin in Δ^X in virtue of our assumptions. Observe that if the dimension of Δ^X is bigger than 3 then all the possible decorations factor through Λ^X_i . We will therefore assume that the dimension is at most 3 otherwise the claim follows directly. Suppose that $\varepsilon=2$ then we can have some Δ^X of dimension 2. In this case our assumptions guarantee that the edge that does not have the vertex i cannot be marked. If $\varepsilon=2$ and the dimension of Δ^X is 3 then it follows that the face that misses the vertex i cannot be thin-scaled. If that face is not lean-scaled then the claim follows immediately. Otherwise our assumptions imply that Δ^X is fully lean scaled the the map $\Lambda^X_i \to \Delta^X$ is a composite of a morphism of type (A1) and a morphism of type (S3). The final case $\varepsilon=3$ is similar and left as an exercise.

We finish the proof by noting that $X, Y \in \mathcal{M}_{\mathcal{A}}^{j}$ it follows that $\sigma_{X} \cap \sigma_{Y} \in Y_{j-1}$ which implies that the order in which the add the simplices is irrelevant. We conclude that each step in the filtration belongs to the weakly saturated hull of morphisms of type (A1) and (S3).

We finish the discussion on dull subsets by giving a right-horn variant of the previous construction.

Definition 2.36. We call a subset $A \subset \mathbb{P}(n)$ right-dull if the following conditions are satisfied

1. \mathcal{A} does not contain \varnothing .

- 2. For every $S \in \mathcal{A}$, $n \notin S$.
- 3. For any $S, T \in \mathcal{A}, S \cap T = \emptyset$.
- 4. For every A-basal subset X we have $u, v \in X$ such that u < v < n.

In this case we call n the pivot point.

Lemma 2.37. Let $A \subset \mathbb{P}(n)$ be a right-dull subset. Let $(\Delta^n)^{\dagger}$ be a marked-biscaled simplex whose thin triangles are degenerate. Suppose that the following conditions holds

- For every A-basal subset X and for every $s, r \in [n]$ such that $s \leq \min(X) < \max(X) \leq r < n$, the triangle $\{s < r < n\}$ is lean, and the edge $r \to n$ is marked.
- Let e be a marked edge in $(\Delta^n)^{\dagger}$ not containing the vertex n. Then e factors through $(S^A)^{\dagger}$.
- Let $\sigma = \{a < b < c\}$ be a lean triangle in $(\Delta^n)^{\dagger}$ not containing the vertex n. Then either σ factors through $(S^A)^{\dagger}$ or $\sigma \cup \{n\}$ is fully lean-scaled and $c \to n$ is marked.

Then $(\mathcal{S}^{\mathcal{A}})^{\dagger} \to (\Delta^n)^{\dagger}$ is in the saturated hull of morphisms of type (A4)

Proof. The argument is nearly identical to the proof of Lemma 2.35.

Lemma 2.38. Let $A \subset \mathbb{P}(n)$ be a right-dull subset. Let $(\Delta^n)^{\dagger} = (\Delta^n, E_n, T_n \subset C_n)$ be a marked-biscaled simplex such that $(\Delta^n)^{\diamond} := (\Delta^n, E_n, \flat \subset C_n)$ satisfies the hypothesis of Lemma 2.37. Suppose that we are given a morphism

$$(\Delta^n, E_n, T_n \subset C_r) \longrightarrow (X, \sharp, T_X \subset \sharp)$$

Then the morphism $(S^A)^{\dagger} \to (\Delta^n)^{\dagger}$ is an **MB**-anodyne morphism over (X, T_X) .

Proof. By Lemma 2.37 we obtain a pushout diagram

$$(\mathcal{S}^{\mathcal{A}})^{\diamond} \longrightarrow (\Delta^{n})^{\diamond}$$

$$\downarrow \qquad \qquad \downarrow$$

$$(\mathcal{S}^{\mathcal{A}})^{\dagger} \longrightarrow P$$

where the top horizontal morphism is **MB**-anodyne. Note P only differs from $(\Delta^n)^{\dagger}$ in its thinscaling. Moreover every lean triangle in P whose image in $(\Delta^n)^{\dagger}$ is thin gets mapped to a thin triangle in (X, T_X) so it can be scaled using a morphism of type (S2).

2.4. Marked scaled simplicial sets

A special case of the model structure of Theorem 2.29 of particular interest occurs when $S = \Delta^0$ is the terminal scaled simplicial set. Then, by [4, Thm 3.39], the resulting model structure on $\operatorname{Set}_{\Delta}^{\mathbf{mb}}$ is Quillen equivalent to the model structure for ∞ -bicategories on $\operatorname{Set}_{\Delta}^{\mathbf{sc}}$. In this case, the data of the two scalings becomes highly redundant — for any fibrant object the two scalings coincide, and heuristically they no longer encode different information.

We can avoid this redundancy by defining a further model structure which includes both markings and scalings, but avoids the redundancies created by a biscaling. The aim of this section is to define this model structure, and relate it to the **MB** model structure.

Definition 2.39. A marked-scaled simplicial set consists of

• A simplicial set X.

- A collection of edges $E_X \subseteq X_1$ containing all degenerate edges. We call the elements of E_X marked edges.
- A collection of triangles $T_X \subseteq X_2$ containing all degenerate triangles. We call the elements of T_X thin triangles.

We denote by $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ the category of marked-scaled simplicial sets. We view this as a $\operatorname{Set}_{\Delta}^+$ -enriched category by defining

$$\operatorname{Hom}_{\operatorname{Set}_{\Delta}^{+}}(X, \operatorname{Set}_{\Delta}^{\mathbf{ms}}(Y, Z)) := \operatorname{Hom}_{\operatorname{Set}_{\Delta}^{\mathbf{ms}}}(X_{\sharp} \times Y, Z)$$

where $X_{\sharp} = (X, E_X, \sharp)$.

Before continuing with the construction of the model structure, we briefly digress to explore the relations between $\operatorname{Set}_{\Delta}^{\mathbf{mb}}$ and $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$. The primary component of our comparison will be the adjunction:

$$\operatorname{Set}_{\Delta}^{\mathbf{ms}} \xrightarrow{D} \operatorname{Set}_{\Delta}^{\mathbf{mb}}$$

where D is given on objects by

$$D: (X, E_X, T_X) \longmapsto (X, E_X, T_X \subseteq T_X)$$

and R is given on objects by

$$R: (Y, E_Y, T_Y \subseteq C_Y) \longmapsto (Y, E_Y, T_Y)$$

We will show that this adjunction becomes a Quillen equivalence once we have equipped $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ with the appropriate model structure.

This model structure itself is constructed exactly analogously to the model structure on $\operatorname{Set}_{\Delta}^{\mathbf{mb}}$. We begin with a set of generating anodyne morphisms:

Definition 2.40. The set of *generating* **MS**-anodyne maps **MS** is the set of maps of marked-scaled simplicial sets consisting of:

(MS1) The inner horn inclusions

$$\left(\Lambda_i^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}\right) \to \left(\Delta^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}\right) \quad , \quad n \geqslant 2 \quad , \quad 0 < i < n;$$

(MS2) The map

$$(\Delta^4, \flat, T) \longrightarrow (\Delta^4, \flat, T \cup \{\Delta^{\{0,3,4\}}, \Delta^{\{0,1,4\}}\})$$

where T is defined as in Definition 2.18, (A2).

(MS3) The set of maps

$$\left(\Lambda_0^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \{\Delta^{\{0,1,n\}}\}\right) \rightarrow \left(\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0, \flat, \{\Delta^{\{0,1,n\}}\}\right) \quad, \quad n \geqslant 2.$$

(MS4) The set of maps

$$\left(\Lambda_n^n, \{ \Delta^{\{n-1,n\}} \}, \{ \Delta^{\{0,n-1,n\}} \} \right) \to \left(\Delta^n, \{ \Delta^{\{n-1,n\}} \}, \{ \Delta^{\{0,n-1,n\}} \} \right) \quad , \quad n \geqslant 2.$$

(MS5) The inclusion of the terminal vertex

$$\left(\Delta^{0},\sharp,\sharp\right) \longrightarrow \left(\Delta^{1},\sharp,\sharp\right)$$

(MS6) The map

$$\left(\Delta^2,\{\Delta^{\{0,1\}},\Delta^{\{1,2\}}\},\sharp\right)\to \left(\Delta^2,\sharp,\sharp\right),$$

(MS7) The map

$$\left(\Delta^{3} \coprod_{\Delta^{\{0,1\}}} \Delta^{0}, \flat, U_{0}\right) \rightarrow \left(\Delta^{3} \coprod_{\Delta^{\{0,1\}}} \Delta^{0}, \flat, \sharp\right)$$

where U_0 is the collection of all triangles except the 0-th face.

(MS8) The map

$$\left(\Delta^3, \{\Delta^{\{2,3\}}\}, U_3\right) \to \left(\Delta^3, \{\Delta^{\{2,3\}}\}, \sharp\right)$$

where U_3 is the collections of all triangles except the 3-rd face.

(MSE) For every Kan complex K, the map

$$(K, \flat, \sharp) \to (K, \sharp, \sharp).$$

We will call a morphism in $\operatorname{Set}^{\mathbf{ms}}_{\Delta}$ MS-anodyne if it lies in the saturated hull of MS.

We can immediately obtain two useful lemmata.

Lemma 2.41. The morphism

$$\left(\Delta^3, \flat, \{\Delta^{\{i-1,i,i+1\}}\} \subset U_i\right) \to \left(\Delta^3, \flat, \{\Delta^{\{i-1,i,i+1\}}\} \subset \sharp\right) \quad, \quad 0 < i < 3,$$

where U_i is the collection of all triangles except i-th face, is MS-anodyne.

Lemma 2.42. The morphism

$$\theta \colon (\Delta^2, \{\Delta^{\{1,2\}}, \Delta^{\{0,2\}}\}, \sharp) \longrightarrow (\Delta^2, \sharp, \sharp)$$

is MS-anodyne.

Proof. The proof follows exactly as in [4, Lem. 3.7].

Finally, in total analogy to the marked biscaled case, we can establish a pushout-product axiom, and thereby a model structure.

Proposition 2.43. Let $f: X \to Y$ be an MS-anodyne morphism in $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$, and let $g: A \to B$ be a cofibration in $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$. The morphism

$$f \wedge g \colon X \times B \coprod_{X \times A} Y \times A \longrightarrow Y \times B$$

is MS-anodyne.

Proof. Every case is, mutatis mutandis, the same as the corresponding case in the proof of [4, Prop. 3.10].

As in the marked-biscaled case, we can immediately define several mapping spaces.

Definition 2.44. Let $\overline{X} := (X, E_X, T_X)$ be a fibrant marked-scaled simplicial set and $\overline{Y} := (Y, E_Y, T_Y)$ any marked-scaled simplicial set. We can define a marked-scaled simplicial set Fun^{ms} $(\overline{Y}, \overline{X})$

via the universal property

$$\operatorname{Hom}_{\operatorname{Set}^{\mathbf{ms}}_{\Lambda}}(\overline{A}, \operatorname{Fun}^{\mathbf{ms}}(\overline{Y}, \overline{X})) \cong \operatorname{Hom}_{\operatorname{Set}^{\mathbf{ms}}_{\Lambda}}(\overline{A} \times \overline{Y}, \overline{X}).$$

It follows from the pushout-product that this is a fibrant marked-scaled simplicial set, and thus that the underlying scaled simplicial set is an ∞ -bicategory. We denote this ∞ -bicategory by $\operatorname{Map}_{\mathbf{ms}}(\overline{Y}, \overline{X})$.

We can similarly define

- A marked simplicial set $\operatorname{Map}_{\mathbf{ms}}^{\mathrm{th}}(\overline{Y}, \overline{X})$ be the full subsimplicial set of $\operatorname{Fun}^{\mathbf{ms}}(\overline{Y}, \overline{X})$ consisting of the thin triangles.
- A simplicial set $\operatorname{Map}_{\mathbf{ms}}^{\simeq}(\overline{Y}, \overline{X})$, which consists of precisely the marked edges in $\operatorname{Map}_{\mathbf{ms}}^{\mathrm{th}}(\overline{Y}, \overline{X})$.

Finally, we can establish the existence of the model structure:

Theorem 2.45. There is a left-proper combinatorial simplicial model category structure on $\operatorname{Set}^{\mathbf{ms}}_{\Delta}$ uniquely characterized by the following properties:

- C) A morphism $f: X \to Y$ in $\operatorname{Set}^{\mathbf{ms}}_{\Delta}$ is a cofibration if and only if it is a monomorphism on underlying simplicial sets.
- F) An object $X \in \operatorname{Set}_{\Delta}^{\mathbf{ms}}$ is fibrant if and only if the unique map $X \to \Delta^0$ has the right lifting property with respect to the morphisms in \mathbf{MS} .

Remark 2.46. It is not hard to see that we can tensor $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ over $\operatorname{Set}_{\Delta}^{+}$ and $\operatorname{Set}_{\Delta}$ in a way compatible with the enrichments provided by $\operatorname{Map}_{\mathbf{ms}}^{\mathrm{th}}(-,-)$ and $\operatorname{Map}_{\mathbf{ms}}^{\sim}(-,-)$, respectively. The latter of these provides the simplicial structure in the preceding proposition.

The weak equivalences in the model structure are precisely those $f: \overline{A} \to \overline{B}$, which satisfy the equivalent conditions for any fibrant marked-scaled simplicial set \overline{X} :

• The induced map

$$\operatorname{Map}_{\mathbf{ms}}(\overline{B}, \overline{X}) \longrightarrow \operatorname{Map}_{\mathbf{ms}}(\overline{A}, \overline{X})$$

is a bicategorical equivalence.

• The induced map

$$\operatorname{Map}_{\mathbf{ms}}^{\operatorname{th}}(\overline{B}, \overline{X}) \longrightarrow \operatorname{Map}_{\mathbf{ms}}^{\operatorname{th}}(\overline{A}, \overline{X})$$

is a weak equivalence of marked simplicial sets.

• The induced map

$$\operatorname{Map}^{\simeq}_{\mathbf{ms}}(\overline{B}, \overline{X}) \longrightarrow \operatorname{Map}^{\simeq}_{\mathbf{ms}}(\overline{A}, \overline{X})$$

is a weak equivalence of Kan complexes.

It is not hard to see that the adjunction $D \dashv R$ can be promoted to a simplicial adjunction. By construction, L preserves cofibrations and R preserves fibrant objects, and thus we see that

Lemma 2.47. The adjunction

$$\operatorname{Set}_{\Delta}^{\mathbf{ms}} \xrightarrow{D} \operatorname{Set}_{\Delta}^{\mathbf{mb}}$$

is a simplicial Quillen adjunction.

Further, we can define an adjunction

$$\operatorname{Set}_{\Delta}^{\mathbf{sc}} \xrightarrow{(-)^{\flat}} \operatorname{Set}_{\Delta}^{\mathbf{ms}}$$

where $G(X, E_X, T_X) = (X, T_X)$.

Lemma 2.48. The adjunction

$$\operatorname{Set}_{\Delta}^{\mathbf{sc}} \xrightarrow{(-)^{\flat}} \operatorname{Set}_{\Delta}^{\mathbf{ms}}$$

is a Quillen adjunction.

Proof. It is immediate that $(-)^{\flat}$ preserves cofibrations. Suppose that $f:(X,T_X)\to (Y,T_Y)$ is a weak equivalence. Let (Z,E_Z,T_Z) be a fibrant object in $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$. It is easy to see that $G(Z,E_Z,T_Z)=(Z,T_Z)$ is a fibrant object in $\operatorname{Set}_{\Delta}^{\mathbf{sc}}$. We can then note that, by definition, there is an isomorphism of mapping scaled simplicial sets

$$\operatorname{Map}_{\mathbf{sc}}((X, T_X), (Z, T_Z)) \cong \operatorname{Map}_{\mathbf{ms}}((X, \flat, T_X), (Z, E_Z, T_Z)).$$

Thus, since f induces a bicategorical equivalence

$$\operatorname{Map}_{\mathbf{sc}}((Y, T_Y), (Z, T_Z)) \to \operatorname{Map}_{\mathbf{sc}}((X, T_X), (Z, T_Z))$$

we see that the map

$$\operatorname{Map}_{\mathbf{ms}}((Y, \flat, T_Y), (Z, E_Z, T_Z)) \to \operatorname{Map}_{\mathbf{ms}}((X, \flat, T_X), (Z, E_Z, T_Z))$$

induced by $(f)^{\flat}$ is also an equivalence. We therefore see that $(f)^{\flat}$ is a weak equivalence in $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$, as desired.

Lemma 2.49. The functor G preserves weak equivalences.

Proof. If, for any ∞ -bicategory (Z, T_Z) , there exists a set E_Z of marked edges for Z such that (Z, E_Z, T_Z) is a fibrant marked-scaled simplicial set, then this follows from the characterization in terms of mapping ∞ -bicategories.

To see that this is the case, let (Z, T_Z) be an ∞ -bicategory. Then Z^{th} is an ∞ -category, and so we can define a marking E_Z on Z by declaring an edge to be marked if it lies in the maximal Kan complex in Z^{th} . From the definition, it is immediate that (Z, E_Z, T_Z) has the extension property with respect to (MS1), (MS2), (MS3), (MS5), (MS6), and (MSE).

It follows from [4, Cor 4.20] and [4, Cor 4.23] that $Z \to \Delta^0$ is a 2-Cartesian fibration in which the strongly Cartesian edges are precisely the equivalences, and so we see that (Z, E_Z, T_Z) has the extension property with respect to (MS4), (MS7), and (MS8) as well.

Lemma 2.50. Given a fibrant marked-scaled simplicial set (Y, E_Y, T_Y) , the full simplicial subset Y^{\simeq} on the marked edges and scaled triangles is a Kan complex.

Proof. It is immediate from the definitions that (Y^{th}, E_Y) is a fibrant marked simplicial set, and the lemma follows.

We now can state and prove the main proposition of this section.

Theorem 2.51. The Quillen adjunctions

$$\operatorname{Set}_{\Delta}^{\mathbf{ms}} \stackrel{D}{\longleftarrow} \operatorname{Set}_{\Delta}^{\mathbf{mb}}$$

and

$$\operatorname{Set}_{\Delta}^{\mathbf{sc}} \xrightarrow{(-)^{\flat}} \operatorname{Set}_{\Delta}^{\mathbf{ms}}$$

are Quillen equivalences.

Proof. By [4, Thm 3.39], the composite adjunction $D \circ (-)^{\flat} \dashv G \circ R$ is a Quillen equivalence. It

thus suffices for us to check that the adjunction $(-)^{\flat} \dashv G$ is a Quillen equivalence. We will check explicitly that the derived adjunction unit and counit are equivalences.

First, let $(X, T_X) \in \operatorname{Set}_{\Delta}^{\mathbf{sc}}$. The derived adjunction unit on (X, T_X) is the composite

$$(X, T_X) \longrightarrow G(X, \flat, T_X) \longrightarrow G((X, \flat, T_X)^{\text{fib}})$$

where the superscript fib denotes fibrant replacement. The first of these maps is the identity (since $G(X, \flat, T_X) = (X, T_X)$) and the latter is the image under G of an equivalence of marked-scaled simplicial sets. By Lemma 2.49, this is an equivalence.

Now, let $(Y, E_Y, T_Y) \in \operatorname{Set}_{\Delta}^{\mathbf{ms}}$ be a fibrant object. The derived adjunction counit on (Y, E_Y, T_Y) is the composite

$$(G(Y, E_Y, T_Y)^{\operatorname{cof}})^{\flat} \longrightarrow G(Y, E_Y, T_Y)^{\flat} \xrightarrow{\eta_Y} (Y, E_Y, T_Y)$$

Since every scaled simplicial set is cofibrant, the first map is an isomorphism, leaving us to check that the usual adjunction counit η_Y is an equivalence. Note that η_Y is simply the inclusion $(Y, \flat, T_Y) \to (Y, E_Y, T_Y)$.

We have a pushout square

$$(Y^{\simeq}, \flat, \sharp) \xrightarrow{\psi} (Y^{\simeq}, \sharp, \sharp)$$

$$\downarrow \qquad \qquad \downarrow$$

$$(Y, \flat, T_Y) \xrightarrow{\eta_Y} (Y, E_Y, T_Y)$$

and, by Lemma 2.50 the morphism ψ is a morphism in MS of type (MSE). Thus, η_Y is MS-anodyne, and is a weak equivalence.

2.4.1. The $\operatorname{Set}_{\Delta}^+$ -enrichment on $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$

We have already constructed a model structure on the category $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ of marked-scaled simplicial sets, and shown that it is a simplicial model category with respect to the mapping spaces $\operatorname{Map}_{\mathbf{ms}}^{\sim}(-,-)$. However, we will need to consider $\operatorname{Set}_{\Delta}^{+}$ -enriched functors in our analysis of the Grothendieck construction. Our aim in this section is therefore to show that our model structure can, additionally, be viewed as $\operatorname{Set}_{\Delta}^{+}$ -enriched. The following lemma constitutes an easy first check in this direction.

Lemma 2.52. The category $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ is powered and tensored over $\operatorname{Set}_{\Delta}^{+}$ via the maps

$$\operatorname{Set}_{\Delta}^{+} \times \operatorname{Set}_{\Delta}^{\mathbf{ms}} \longrightarrow \operatorname{Set}_{\Delta}^{\mathbf{ms}}$$
$$(K, X) \longmapsto K_{\sharp} \times X$$

and

$$[-,-]: \operatorname{Set}_{\Delta}^{+} \times \operatorname{Set}_{\Delta}^{\mathbf{ms}} \longrightarrow \operatorname{Set}_{\Delta}^{\mathbf{ms}}$$

$$(K,X) \longmapsto \operatorname{Fun}^{\mathbf{ms}}(K_{\sharp},X)$$

The tensoring and powering is compatible with the mapping spaces $\operatorname{Map}_{\mathbf{ms}}^{\mathrm{th}}(-,-)$.

Our aim throughout the rest of the section will be to show that the tensoring is a left Quillen bifunctor. We will follow the strategy of [9], showing first that the model structure on $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ is a Cisinski-Olschok model structure (as with $\operatorname{Set}_{\Delta}^{\mathbf{sc}}$ in [10]), and then using testing pushout-products with the concomitant interval objects.

We first show that the model structure on $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ is Cartesian-closed. This will follow immediately from Proposition 2.43 and the following

Lemma 2.53. Let $f: X \to Y$ and $g: A \to B$ be two weak equivalences in $\operatorname{Set}_{\Delta}^{\mathbf{mb}}$, then the product

$$f \times g \colon X \times A \longrightarrow Y \times B$$

is a weak equivalence.

Proof. Precisely the same argument as in [23, Lemma 4.2.6] allows us to reduce to the case of the morphism

$$Y \times A \longrightarrow Y \times B$$

where Y, A, and B are all fibrant objects. By the characterization of fibrant objects, this morphism is a weak equivalence if and only if the morphism on underlying scaled simplicial sets is an equivalence, which follows from loc. cit.

Corollary 2.54. For any cofibrations $f: X \to Y$ and $g: A \to B$, the pushout-product

$$f \wedge g \colon Y \times A \coprod_{X \times A} X \times B \longrightarrow Y \times B$$

is an equivalence if one of f or g is.

Proof. We can use the small object argument to factor f as

$$X \xrightarrow{h} Z \xrightarrow{k} Y$$

where h is MS-anodyne. Consequently, k is a weak equivalence. We consider the diagram

It follows from the lemma that the bottom horizontal arrow is a weak equivalence, and the top horizontal arrow is the induced map on homotopy colimits by a natural weak equivalence. from Proposition 2.43, it follows that the right-hand morphism is an equivalence, and the corollary follows.

Corollary 2.55. The model structure on $\operatorname{Set}^{\mathbf{ms}}_{\Delta}$ is Cartesian-closed.

We now wish to show that the Cisinski-Olschok model structure on $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ with interval Δ^0 II $\Delta^0 \to (\Delta^1)^\sharp_{\sharp}$ and generating anodyne maps the **MS**-anodyne maps is, in fact the model structure constructed in our previous section. We first note that, since one of the morphism $\Delta^0 \to (\Delta^1)^\sharp_{\sharp}$ is **MS**-anodyne, it follows that both such morphisms are trivial cofibrations.

Definition 2.56. We write $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})_{\mathrm{CO}}$ for the Cisinski-Olschok model structure on $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ with interval $\Delta^0 \coprod \Delta^0 \to (\Delta^1)^\sharp_{\sharp}$, and generating set of anodyne morphisms the set of **MB**-anodyne morphisms.

For ease, we will write $(Set^{\mathbf{ms}}_{\Delta})_{AH}$ for the model structure previously defined

Proposition 2.57. The two model structures $(Set^{\mathbf{ms}}_{\Delta})_{CO}$ and $(Set^{\mathbf{ms}}_{\Delta})_{AH}$ coincide.

Proof. It will suffice to show that the fibrant objects coincide. By construction, every fibrant object of $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})_{CO}$ is a fibrant object of $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})_{AH}$. However, since $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})_{AH}$ is a Cartesian-closed model category, and the interval object for $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})_{CO}$ is a cylinder in $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})_{AH}$, every anodyne map in $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})_{CO}$ is a trivial cofibration in $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})_{AH}$. Thus every fibrant object of $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})_{AH}$ is a fibrant object of $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})_{CO}$.

As a consequence, we will now drop the unwieldy subscript notation for the model structure on $\operatorname{Set}^{\mathbf{ms}}_{\Delta}$. We can now prove the following.

Proposition 2.58. The model category $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ is a $\operatorname{Set}_{\Delta}^{+}$ -enriched model category.

Proof. We need only show that the tensoring satisfies the pushout-product axiom, i.e., that for cofibrations $f: K \to S$ in $\operatorname{Set}_{\Delta}^+$ and $g: X \to Y$ in $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$, the pushout-product $f \wedge g$ is a trivial cofibration that either f or g is. Since both model structures are Cisinski-Olschok model structures, it suffices to test generating monomorphisms against the two interval inclusions and against the generating anodyne morphisms.

It is immediate from Proposition 2.43 that if f (resp. g) is marked (resp. \mathbf{MS}) anodyne, then $f \wedge g$ is a trivial cofibration. It remains for us to test the cases when f is $\{0\} \to (\Delta^1)^{\sharp}$ or $\{1\} \to (\Delta^1)^{\sharp}$, and the cases when g is $\{0\} \to (\Delta^1)^{\sharp}$ or $\{1\} \to (\Delta^1)^{\sharp}$.

However, since the morphisms $\{0\} \to (\Delta^1)^\sharp_\sharp$ or $\{1\} \to (\Delta^1)^\sharp_\sharp$ are trivial cofibrations, and the model structure on $\operatorname{Set}^{\mathbf{ms}}_\Delta$ is Cartesian-closed, this follows immediately.

3. The bicategorical Grothendieck construction

Our first step towards an ∞ -bicategorical Grothendieck construction is defining the functors which will realize the desired equivalence. These definitions will constitute an upgrade of the straightening and unstraightening constructions of [21, Section 3.2] to the more highly decorated setting of marked-biscaled simplicial sets and marked-scaled simplicial sets. These functors will define a Quillen equivalence of model categories between $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$ and a model category we now define.

Definition 3.1. Let \mathcal{C} be a $\operatorname{Set}_{\Delta}^+$ -category. We denote by $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathcal{C}}$ the category of $\operatorname{Set}_{\Delta}^+$ -enriched functors and natural transformations. We endow the category of enriched functors with the projective model structure (See, e.g., [21, A.3.3.2]).

Definition 3.2. Let (Y, E_Y, T_Y) be a marked scaled simplicial set. We define a scaled simplicial set which we denote $(Y^{\triangleright}, T_{Y^{\triangleright}})$ whose underlying simplicial set is given by $Y^{\triangleright} = Y * \Delta^0$ and whose non-degenerate scaled simplices are either those that factor through Y or those of the form $f * \mathrm{id}_{\Delta^0}$ where $f : \Delta^1 \to Y$ belongs to E_Y .

Remark 3.3 (Important convention). Let $(X, M_X, T_X \subseteq C_X)$ be an **MB** simplicial set. By the underlying scaled simplicial set, we will mean the scaled simplicial set (X, T_X) .

Remark 3.4 (Notation for ops). Given a simplicial set X with any decoration (marking, scaling, etc.), we will denote by X^{op} the opposite simplicial set with the same decoration.

Given an enriched category \mathcal{C} (a $\operatorname{Set}_{\Delta}^+$ -enriched category, a 2-category, etc.), we will denote $\mathcal{C}^{\operatorname{op}}$ the enriched category with the same objects and $\mathcal{C}^{\operatorname{op}}(x,y)=\mathcal{C}(y,x)$. In the specific case of a 2-category \mathbb{C} , we will occasionally write $\mathbb{C}^{(\operatorname{op},-)}$ to denote $\mathbb{C}^{\operatorname{op}}$. We will only rarely make use of the 2-morphism dual $\mathbb{C}^{(-,\operatorname{op})}$.

We now provide the underlying left Quillen functor of our bicategorical Grothendieck construc-

Construction 3.5. Fix a scaled simplicial set $S \in \operatorname{Set}_{\Delta}^{\mathbf{sc}}$ and a functor of $\operatorname{Set}_{\Delta}^{+}$ -enriched categories $\phi : \mathfrak{C}^{\mathbf{sc}}[S] \to \mathfrak{C}$. Let $p : X \to S$ be an object of $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$. We define a scaled simplicial set X_S via the pushout diagram

$$\begin{array}{ccc} X & \longrightarrow & X^{\triangleright} \\ \downarrow & & \downarrow \\ S & \longrightarrow & X_S \end{array}$$

We generically denote both the cone point of X^{\triangleright} and its image in X_S by *. We then define a $\operatorname{Set}_{\Delta}^+$ -enriched category

$$X_{\phi} := \mathfrak{C} \coprod_{\mathfrak{C}^{\mathrm{sc}}[S]} \mathfrak{C}^{\mathrm{sc}}[X_S].$$

Note that this is equivalently the pushout

$$\begin{array}{ccc} \mathfrak{C}^{\mathrm{sc}}[X] & \longrightarrow \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}] \\ & & \downarrow \\ \phi \circ \mathfrak{C}[p] \downarrow & & \downarrow \\ \mathbb{C} & \longrightarrow X_{\phi} \end{array}$$

of $\operatorname{Set}_{\Lambda}^+$ -enriched categories.

Applying the enriched Yoneda embedding on the cone point *, this provides a $\operatorname{Set}_{\Delta}^+$ -enriched functor

$$\operatorname{St}_{\phi}^+(X) \colon \mathcal{C}^{\operatorname{op}} \longrightarrow \operatorname{Set}_{\Delta}^+$$

 $s \longmapsto X_{\phi}(s,*).$

We promote this functor to a $\operatorname{Set}_{\Delta}^+$ -enriched functor

$$\operatorname{St}_{\phi}(X) \colon \mathfrak{C}^{\operatorname{op}} \longrightarrow \operatorname{Set}_{\Delta}^{\mathbf{ms}}$$

by equipping its values on objects with a scaling.

After a single, fairly ad-hoc definition, we are able to do this in a highly functorial way. The ad-hoc definition will be a promotion of $\mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}]$ to a $\mathrm{Set}_{\Delta}^{\mathbf{ms}}$ -enriched category, such that the subcategory $\mathfrak{C}^{\mathrm{sc}}[X] \subset \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}]$ has all mapping spaces maximally scaled. We will denote the resulting $\mathrm{Set}_{\Delta}^{\mathbf{ms}}$ -enriched category $\mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}]_{\dagger}$. More generally, we will denote scalings on the mapping spaces of a marked-simplicially enriched category \mathfrak{C} using subscripts, e.g. \mathfrak{C}_{\sharp} for maximally marked mapping spaces.

We will define the scaling on $\mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}]$ in three steps:

1. We define the scaling

$$\mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}]_{\dagger}(s,t) := \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}](s,t)_{\sharp}$$

for $s, t \in X$.

2. We define an auxiliary scaling $P_{X^{\triangleright}}^s$ on each marked simplicial set $\mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}](s,*)$. iven a map $\sigma \colon \Delta^n \longrightarrow X$, we can pass to the associated n+1-simplex $\sigma \star \mathrm{id}_0 \colon \Delta^{n+1} \longrightarrow X^{\triangleright}$ and obtain a map of simplicial sets

$$\mathfrak{C}^{\mathrm{sc}}[\Delta^{n+1}](0,n+1) \longrightarrow \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}](\sigma(0),*).$$

Each 2-simplex in $\mathfrak{C}^{\mathrm{sc}}[\Delta^{n+1}](0, n+1)$ is of the form

$$S_0 \cup \{n+1\} \subset S_1 \cup \{n+1\} \subset S_2 \cup \{n+1\}$$

where $S_i \subseteq [n]$ contains 0. We declare the image of such a 2-simplex to be scaled in $\mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}](s,*)$ precisely when either

- $\max(S_i) = \max(S_i)$ for some $i, j \in \{0, 1, 2\}$; or
- the simplex σ is lean in X (i.e., lies in C_X) and the 2-simplex is $03 \to 013 \to 0123$.

The auxiliary scaling $P_{X^{\triangleright}}^s$ then consists of all such 2-simplices.

3. We extend the scaling $P_{X^{\triangleright}}^s$ by functoriality. That is, we declare a 2-simplex $\sigma: \Delta^2 \to \mathfrak{C}[X^{\triangleright}](s,*)$ to be scaled if there is a t in X and a 2-simplex

$$\theta = (\theta_1, \theta_2) \colon \Delta^2 \longrightarrow \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}](s, t) \times \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}](t, *)$$

such that $\theta_2 \circ \theta_1 = \sigma$, where $\theta_2 \in P_{X^{\triangleright}}^s$. We would like to stress to the reader that this also adds scaled 2-simplices in the case where θ_2 is degenerate.

We can then define a $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ -enriched variant of X_{ϕ} to be the pushout of $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ -enriched categories

$$\mathfrak{C}^{\mathrm{sc}}[X]_{\sharp} \longrightarrow \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}]_{\dagger} \\
\downarrow^{\phi \circ \mathfrak{C}[p]} \qquad \qquad \downarrow^{\chi_{\phi}} \\
\mathfrak{C}_{\sharp} \longrightarrow \overline{X_{\phi}}$$

Unwinding the definitions, we see that a 2-simplex $\sigma: \Delta^2 \to \overline{X_{\phi}}(s,*)$ is scaled if and only if it satisfies the following condition:

• There is a $t \in X_{\phi}$ and a 2-simplex

$$\theta = (\theta_1, \theta_2) : \Delta^2 \longrightarrow \overline{X_{\phi}}(s, t) \times \overline{X_{\phi}}(t, *)$$

such that (1) $\sigma = \theta_2 \circ \theta_1$, and, (2) θ_2 is either in the image of an element of $P_{X^{\triangleright}}^s$ or is degenerate.

The bicategorical straightening of X is then the restriction of the $\operatorname{Set}^{\mathbf{ms}}_{\Delta}$ -enriched Yoneda embedding:

$$\operatorname{St}_{\phi}(X) \colon \operatorname{\mathcal{C}}^{\operatorname{op}} \longrightarrow \operatorname{Set}_{\Delta}^{\mathbf{ms}}$$
$$s \longmapsto \overline{X_{\phi}}(s, *).$$

A priori, this is an $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ -enriched functor. However, since we required the mapping spaces in \mathcal{C} to be maximally scaled, this formula in fact defines an $\operatorname{Set}_{\Delta}^+$ -enriched functor. This construction then yields a functor

$$\mathbb{S}t_{\phi} \colon (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S} \longrightarrow (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathcal{C}^{\operatorname{op}}}$$

which we call the (bicategorical) straightening functor.

Notation. We will denote by $St_S(X)$ the special case in which $\phi: \mathfrak{C}^{\mathrm{sc}}[S] \to \mathfrak{C}^{\mathrm{sc}}[S]$ is the identity.

Remark 3.6. In line with the philosophy of [29], there should be a model for $(\infty, 3)$ -categories on the category of simplicial sets with decorations on 1-, 2-, and 3-simplices. The ad-hoc construction of the Set^{ms}_{Δ}-enriched category $\overline{X_{\phi}}$ above seems likely to fit into some — as-yet-undefined — $(\infty, 3)$ -categorical version of the rigidification functor, which turns decorated 3-simplices in scaled 2-simplices in the corresponding mapping space.

Remark 3.7. Given a 2-Cartesian fibration $p: X \to S$, we note that if *every* triangle in X is lean, the map $\operatorname{St}_S(X)(i) \to \operatorname{St}_S(X)(i)_{\sharp}$ is an equivalence of marked-scaled simplicial sets. More generally, we obtain a diagram

$$\begin{array}{ccc}
(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S} & \xrightarrow{\mathbb{S}t_{S}} (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[S]^{\mathrm{op}}} \\
(-)_{T \subset \sharp} & & \uparrow (-)_{\sharp} \\
(\operatorname{Set}_{\Delta}^{\mathbf{ms}})_{/S} & \xrightarrow{\operatorname{St}^{+}} (\operatorname{Set}_{\Delta}^{+})^{\mathfrak{C}^{\mathrm{sc}}[S]^{\mathrm{op}}}
\end{array}$$

which commutes up to natural weak equivalence.

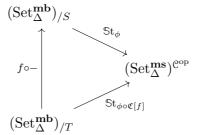
While we will not formalize this statement here, there should be a model structure on $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})_{/S}$ modeling ∞ -bicategories fibred in $(\infty,1)$ -categories, such that St^+ becomes a left Quillen equivalence to the projective model structure. The diagram above would then represent the restriction of our straightening-unstraightening equivalence to this special case.

3.1. First properties

Before proceeding to the technical nitty-gritty of the Quillen equivalences, we establish some basic properties of the straightening functor.

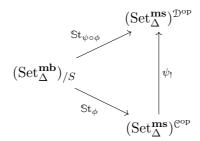
Proposition 3.8. Let $S \in \operatorname{Set}_{\Delta}^{\operatorname{sc}}$ and let $\phi : \mathfrak{C}^{\operatorname{sc}}[S] \to \mathfrak{C}$ be a $\operatorname{Set}_{\Delta}^+$ -enriched functor. Then the following hold

- 1. The straightening functor St_{ϕ} preserves colimits.
- 2. (Base change for scaled functors) Given a morphism of scaled simplicial sets $f: T \to S$ there a diagram



which commutes up to natural isomorphism of functors.

3. (Base change for $\operatorname{Set}_{\Delta}^+$ -functors) Given a $\operatorname{Set}_{\Delta}^+$ -enriched functor $\psi: \mathcal{C} \to \mathcal{D}$ there is a diagram



which commutes up to natural isomorphism of functors.

Proof. All three statements hold on the level of $\operatorname{St}_{\phi}^+$, and so the proof amounts to checking scalings. We prove (1), and leave the other two statements to the reader.

It is follows from the definition that

$$\operatorname{St}_{\phi}^{+} \colon (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S} \longrightarrow (\operatorname{Set}_{\Delta}^{+})^{\mathcal{C}^{\operatorname{op}}}$$

preserves colimits. Since colimits in functor categories are computed pointwise, it will thus suffice to show that, given a diagram

$$D: I \longrightarrow (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$$

the scalings on $\operatorname{colim}_I \operatorname{St}_{\phi}(D(i))$ and $\operatorname{St}_{\phi}(\operatorname{colim}_I D(i))$ coincide. Indeed, applying the universal property, it will suffice to show that the map

$$\operatorname{St}_{\phi}(\operatorname{colim}_{I}D(i)) \longrightarrow \operatorname{colim}_{I}\operatorname{St}_{\phi}(D(i))$$

which is the identity on underlying marked simplicial sets preserves the scalings.

Fix $s \in \mathcal{C}$, we will first show that the map

$$f_s : (\operatorname{St}_{\phi}(\operatorname{colim}_I D(i)))(s) \longrightarrow (\operatorname{colim}_I \operatorname{St}_{\phi}(D(i)))(s)$$

preserves the scalings inherited from $P^s_{X^{\triangleright}}$. To this end, suppose given a scaled simplex σ in $P^s_{(\operatorname{St}_{\phi}(\operatorname{colim}_I D(i)))_S}$ which does not come from a lean simplex in the colimit. Tracing through the definition, we note that there must be a simplex $\eta: \Delta^n \to \operatorname{colim}_I(D(i))$ and a simplex $\mu:=\{S_0 \cup \{n+1\} \to S_1 \cup \{n+1\} \to S_2 \cup \{n+1\}\}$ in $\mathfrak{C}^{\operatorname{sc}}[\Delta^{n+1}](0,n+1)$ with $\max(S_i) = \max(S_j)$ for

some i, j = 0, 1, 2 such that σ is the image of μ under the canonical map

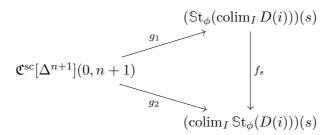
$$g_1: \mathfrak{C}^{\mathrm{sc}}[\Delta^{n+1}](0, n+1) \longrightarrow (\operatorname{St}_{\phi}(\operatorname{colim}_I D(i)))(s)$$

is not scaled.

By the construction of colimits in simplicial sets, this means that there is an $k \in I$ and a simplex $\hat{\eta}:\Delta^n\to D(k)$ such that η factors through the canonical map $D(k)\to\operatorname{colim}_I D(i)$ as $\hat{\eta}$. We can then note that $\hat{\eta}$ will yield a map

$$g_2 \colon \mathfrak{C}^{\mathrm{sc}}[\Delta^{n+1}](0,n+1) \longrightarrow \mathbb{S}\mathrm{t}_{\phi}(D(k))(s) \longrightarrow (\operatorname*{colim}_{I} \mathbb{S}\mathrm{t}_{\phi}(D(i)))(s)$$

such that the diagram



commutes. We thus see that $g_2(\mu) = f_s(g_1(\mu)) = f_s(\sigma)$ is scaled, as desired. The same argument

holds, mutatis mutandis, for $\sigma \in P^s_{(\operatorname{St}_\phi(\operatorname{colim}_I D(i)))_S}$ coming from a lean 2-simplex in the colimit. We can now easily check that the full scalings $T^s_{X_S}$ are preserved by f_s by simply noting that the diagram

$$\begin{array}{ccc}
\mathbb{C}(s',s) \times (\operatorname{St}_{S}^{+}(\operatorname{colim}_{I}D(i)))(s) & \xrightarrow{\circ} & (\operatorname{St}_{S}^{+}(\operatorname{colim}_{I}D(i)))(s') \\
& f_{s} \downarrow & \downarrow f_{s'} \\
\mathbb{C}(s',s) \times (\operatorname{colim}_{I}\operatorname{St}_{S}(D(i)))(s) & \xrightarrow{\circ} & (\operatorname{colim}_{I}\operatorname{St}_{S}(D(i)))(s')
\end{array}$$

commutes.

To show (2) and (3), we again note that the statements are immediate if we replace St_{ϕ} with $\operatorname{St}_{\phi}^{+}$ (cf. [23, Rmk 3.5.16] and [21, Prop 3.2.1.4]). A similar check to the above assures us that the scalings coincide.

Remark 3.9. Note that, in the case where we consider ϕ to be the identity on $\mathfrak{C}^{\mathrm{sc}}[S]$ and are given a morphism $f: T \to S$, combining (2) and (3) in Proposition 3.8 yields a diagram

$$(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S} \xrightarrow{\operatorname{St}_{S}} (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\operatorname{sc}}[S]^{\operatorname{op}}}$$

$$f \circ - \uparrow \qquad \qquad \uparrow \mathfrak{C}^{\operatorname{sc}}[f]!$$

$$(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/T} \xrightarrow{\operatorname{St}_{T}} (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\operatorname{sc}}[T]^{\operatorname{op}}}$$

which commutes up to natural isomorphism.

Corollary 3.10. Let S be an scaled simplicial set and $\phi : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathfrak{C}$ a Set_{Λ}^+ -enriched functor. Then the straightening functor St_{ϕ} has a right adjoint

$$\mathbb{U}_{n_{\phi}} \colon \left(\operatorname{Set}_{\Delta}^{\mathbf{ms}} \right)^{\mathcal{C}^{\operatorname{op}}} \longrightarrow \left(\operatorname{Set}_{\Delta}^{\mathbf{mb}} \right)_{/S}$$

which we call the (bicategorical) unstraightening functor.

Proof. This follows from the first part in Proposition 3.8 using the adjoint functor theorem. Let Δ^n_{\flat} denote the minimally scaled *n*-simplex and consider $(\Delta^n)^{\flat}_{\flat} = (\Delta^n, \flat, \flat)$ as an object of $(\operatorname{Set}^{\mathbf{mb}}_{\Delta})_{/\Delta^n_{\flat}}$ via the identity map. To ease the notation we will denote the straightening of this object as $\operatorname{St}_{\Delta^n}(\Delta^n)$.

Definition 3.11. Let $n \ge 0$ and $0 \le s \le n$. We denote by $L^n(s)$ the poset of subsets $S \subseteq [n]$ such that $\min(S) = s$ ordered by inclusion. Let $\sigma: S_0 \subseteq S_1 \subseteq S_2$ be a 2-simplex in the (nerve of) $L^n(s)$ and denote $s_i = \max(S_i)$ for i = 0, 1, 2. We say that σ is *thin* if there exists a pair of indices i, j such that $s_i = s_j$. We endow $L^n(s)$ with a scaling given by thin simplices and with the minimal marking. The resulting marked scaled simplicial set will be denoted by $\mathcal{L}^n_b(s)$

Lemma 3.12. Let $n \ge 0$ and $0 \le s \le n$. Then there is an isomorphism

$$\operatorname{St}_{\Delta^n_{\flat}}(\Delta^n)(s) \xrightarrow{\simeq} \mathcal{L}^n_{\flat}(s)$$

of marked scaled simplicial sets

Proof. Immediate from unraveling the definitions.

Definition 3.13. Let $n \ge 0$ and consider a **MB** simplicial set $\Delta_T^n := (\Delta^n, \flat, \flat \subseteq T)$ for some scaling T. Given $0 \le s \le n$ we define a new scaling on $L^n(s)$ (see Definition 3.11) by declaring a 2-simplex $S_0 \subseteq S_1 \subseteq S_2$ if and only if the simplex defined by $\max(S_0) \le \max(S_1) \le \max(S_2)$ is lean in Δ_T^n . We denote the resulting scaled simplicial set by $\mathcal{L}_T^n(s)$.

Lemma 3.14. Let $\Delta_T^n := (\Delta^n, \flat, \flat \subseteq T)$ and denote by $\operatorname{St}_{\Delta_{\flat}^n}(\Delta_T^n)$ the straightening of the map $\Delta_T^n \to \Delta_{\flat}^n$. Then for every $0 \leqslant s \leqslant n$ the canonical map

$$\operatorname{St}_{\Delta_{\mathsf{b}}^n}(\Delta_T^n)(s) \longrightarrow \mathcal{L}_T^n(s)$$

is MS-anodyne.

Proof. The existence of the morphism is clear from the definitions. Suppose that we are given a thin 2-simplex $\sigma: S_0 \subseteq S_1 \subseteq S_2$ in $\mathcal{L}^n_T(i)$. As before, we adopt the convention that $s_i := \max(S_i)$. We will show that σ can be scaled by taking pushouts along **MS**-anodyne morphisms. First let us consider the 3-simplex

$$\theta: S_0 \subseteq S_0 \cup \{s_1\} \subseteq S_0 \cup \{s_1, s_2\} \subseteq S_2$$

We immediately observe that all of its faces are scaled in $\operatorname{St}_{\Delta^n_{\flat}}(\Delta^n)_T(s)$ except the face missing 2. It follows we can scale the remaining face using a pushout along a **MS**-anodyne map of the type described in Lemma 2.41. Now we consider another 3-simplex

$$\rho: S_0 \subseteq S_0 \cup \{s_1\} \subseteq S_1 \subseteq S_2$$

Again we observe that all of its faces are scaled except possibly the face missing 1 which is precisely σ . The conclusion easily follows from Lemma 2.41

Let S_{\sharp} be a scaled simplicial set and assume every triangle is thin. Denote by S its underlying simplicial set and let $(\operatorname{Set}_{\Delta}^+)_{/S}$ denote the category of marked simplicial sets over S. We define a functor

$$\iota \colon (\operatorname{Set}_{\Delta}^+)_{/S} \longrightarrow (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}, \ (X, E_X) \longmapsto (X, E_X, \sharp)$$

We view the ∞ -categorical straightening functor St_S (see 3.2.1 in [21]) as a functor with values $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\operatorname{sc}}[S]^{\operatorname{op}}}$ by maximally scaling the values of $\operatorname{St}_S X(s)$.

Proposition 3.15. There exists a natural transformation

$$\varepsilon \colon \mathbb{S} \mathfrak{t}_S \circ \iota \longrightarrow \mathbb{S} \mathfrak{t}_S$$

which is objectwise a weak equivalence of marked scaled simplicial sets.

Proof. The existence of the natural transformation is automatic since both functors only differ on the scaling. It is clear that both functors preserve colimits and that they satisfy base change to respect to morphisms of simplicial sets $S \to T$. In addition, it is routine to verify that both functors respect cofibrations. An standard argument then shows that it suffices to check that the natural transformation is an equivalence (1) when $S = (\Delta^n)^{\flat}$ with $n \ge 0$ and $X \to S$ is the identity morphism, and (2) on $(\Delta^1)^{\sharp} \to \Delta^1$ when $S = \Delta^1$. This is a direct consequence of Lemma 3.14. \square

We conclude this section with a first step towards showing that the bicategorical straightening is left Quillen.

Proposition 3.16. Let S be a scaled simplicial set and let $\phi : \mathfrak{C}^{\mathrm{sc}}[S] \to \mathcal{C}$ be a Set_{Δ}^+ -enriched functor. Then the straightening functor

$$\mathbb{S}_{t_{\phi}}: (\operatorname{Set}_{\Lambda}^{\mathbf{mb}})_{/S} \longrightarrow (\operatorname{Set}_{\Lambda}^{\mathbf{ms}})^{\mathfrak{C}[\mathcal{C}]^{\operatorname{op}}}$$

preserves cofibrations.

Proof. The generators of the class of cofibrations of marked biscaled simplicial sets are given by

(C1)
$$(\partial \Delta^n, \flat, \flat) \rightarrow (\Delta^n, \flat, \flat).$$

(C2)
$$(\Delta^1, \flat, \flat) \rightarrow (\Delta^1, \sharp, \flat).$$

(C3)
$$(\Delta^2, \flat, \flat) \rightarrow (\Delta^2, \flat, \flat \subset \sharp).$$

(C4)
$$(\Delta^2, \flat, \flat \subset \sharp) \to (\Delta^2, \flat, \sharp).$$

Note that (C4) and (S2) are the same morphism. Therefore using standard arguments it will suffice to check our claim on those generators.

Let $i:A\to B$ be a cofibration. As stated above it will suffice to check in the case where i is one of the generating cofibrations. Furthermore we can use Proposition 3.8 to reduce to the case where S is the underlying scaled simplicial set of B, and ϕ is id: $\mathfrak{C}^{\mathrm{sc}}[S] \to \mathfrak{C}^{\mathrm{sc}}[S]$. The result follows from a straightforward computation.

3.2. Products and tensoring

Before we can proceed to proving that the straightening-unstraightening adjunction is a Quillen equivalence (indeed, before we can prove the straightening is left Quillen), we need to establish the relation of the straightening to the $\operatorname{Set}_{\Delta}^+$ -tensoring. We will prove this as a corollary of a more general result — on products of $\operatorname{\mathbf{MB}}$ simplicial sets — which will be of use to us in the sequel.

Let $A, B \in \operatorname{Set}_{\Delta}^{\mathbf{sc}}$ and consider a pair of objects $X_A \in (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/A}$, $X_B \in (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/B}$ giving rise to $X \in (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/A \times B}$ then we can form a pushout diagram

$$\begin{array}{ccc} \mathfrak{C}^{\mathrm{sc}}[X] & \longrightarrow & \mathfrak{C}^{\mathrm{sc}}[X^{\triangleright}] \\ & & \downarrow^{\phi} & & \downarrow \\ \mathfrak{C}^{\mathrm{sc}}[A] \times \mathfrak{C}^{\mathrm{sc}}[B] & \longrightarrow & X_{A,B} \end{array}$$

where the left-most vertical morphism is the composite $\mathfrak{C}^{\mathrm{sc}}[X] \to \mathfrak{C}^{\mathrm{sc}}[A \times B] \to \mathfrak{C}^{\mathrm{sc}}[A] \times \mathfrak{C}^{\mathrm{sc}}[B]$. Let

$$\operatorname{St}_A X_A \boxtimes \operatorname{St}_B X_B : \mathfrak{C}^{\operatorname{sc}}[A]^{\operatorname{op}} \times \mathfrak{C}^{\operatorname{sc}}[B]^{\operatorname{op}} \to \operatorname{Set}_{\Delta}^{\operatorname{\mathbf{ms}}}$$

be the pointwise product of $\mathbb{S}_{t_A} X_A$ and $\mathbb{S}_{t_B} X_B$ and observe there is a canonical natural transformation

$$\varepsilon_X : \operatorname{St}_{\phi} X \Rightarrow \operatorname{St}_A X_A \boxtimes \operatorname{St}_B X_B$$

We will prove the following theorem:

Theorem 3.17. The map $\varepsilon_X : \operatorname{St}_{\phi} X \Rightarrow \operatorname{St}_A(X_A) \boxtimes \operatorname{St}_B(X_B)$ is a pointwise weak equivalence.

Before proceeding with the proof of the theorem we need to do some preliminary work. First we will do a careful study of the case where $A = (\Delta^n, \flat)$ and $B = (\Delta^k, \flat)$, $X_A = (\Delta^n, \flat, \flat)$ and $X_B = (\Delta^k, \flat, \flat)$. We will assume that the maps $X_A \to A$ and $X_B \to B$ are the identity on the underlying scaled simplicial sets. In this particular situation we will denote $\operatorname{St}_{\phi} X(i,j) := \mathbb{P}^{n,k}_{(i,j)}$ and $\operatorname{St}_{\Delta^n} \Delta^n(i) \times \operatorname{St}_{\Delta^k} \Delta^k(j) := \mathbb{S}^{n,k}_{(i,j)}$.

Definition 3.18. Let $n, k \ge 0$ and let $i \in [n], j \in [k]$. We define marked scaled simplicial set $\mathbb{E}^{n,k}_{(i,j)}$ whose underlying simplicial set is given by $\mathfrak{C}[(\Delta^n \times \Delta^k)^{\triangleright}]((i,j),*)$. To define the marking and the scaling we construct a morphism

 $\xi_{(i,j)}^{n,k} \colon \mathbb{E}_{(i,j)}^{n,k} \longrightarrow \mathbb{S}_{(i,j)}^{n,k}$

an equip $\mathbb{E}^{n,k}_{(i,j)}$ with the induced marking and scaling. Recall that objects of $\mathbb{E}^{n,k}_{(i,j)}$ are given by a chain or sequence of inequalities $(a_0,b_0)<(a_1,b_1)<\cdots(a_\ell,b_\ell)$ where $a_i\in[n]$ and $b_i\in[n]$ for $i=0,\ldots,\ell$ and with the property that $(a_0,b_0)=(i,j)$. We will use the notation $C=\{(a_i,b_i)\}_{i=0}^\ell$. A morphism between chains $C_1\to C_2$ is simply given by an inclusion $C_1\subset C_2$ which we call a refinement of the chain C_1 . Then we define $\xi(C)=(S_a,S_b)$ where $S_a=\{a_0,a_1,\ldots,a_\ell\}$ and similarly for S_b .

Remark 3.19. It is immediate to see that the map $\xi_{(i,j)}^{n,k}$ constructed before factors as

$$\mathbb{E}^{n,k}_{(i,j)} \longrightarrow \mathbb{P}^{n,k}_{(i,j)} \longrightarrow \mathbb{S}^{n,k}_{(i,j)}$$

where the second morphism is the component of the natural transformation ε_X at the object (i,j) and the first morphism is a canonical collapse map. We will denote the first morphism by $\pi_{(i,j)}^{n,k}$ and the second morphism by $\varepsilon_{(i,j)}^{n,k}$.

Definition 3.20. Let $C \in \mathbb{E}_{(i,j)}^{n,k}$ be a chain denoted by $C = \{(a_i,b_i)\}_{i=0}^{\ell}$. We set $|C| = \ell$ and we call it the *length* of the chain.

Definition 3.21. Let $C \in \mathbb{E}^{n,k}_{(0,0)}$. We define \mathcal{E}_C to be the full subposet (with the induced marking and scaling) of $\mathbb{E}^{n,k}_{(0,0)}$ consisting of those chains K contained in C.

Definition 3.22. Let $C \in \mathbb{E}^{n,k}_{(0,0)}$ be a chain. We say that $K \in \mathcal{E}_C$ is a *rigid chain* if there is no marked morphism in \mathcal{E}_C with source K. We denote the by \mathcal{E}^r_C the full subposet of \mathcal{E}_C on rigid chains.

Lemma 3.23. Let $C \in \mathbb{E}_{(0,0)}^{n,k}$ be a chain and denote by \mathcal{U}_C the image of the morphism $\mathcal{E}_C \to \mathbb{S}_{(0,0)}^{n,k}$. Then $\xi_{(0,0)}^{n,k}$ induces an isomorphism of marked scaled simplicial sets

$$\xi_C^r: \mathcal{E}_C^r \stackrel{\cong}{\longrightarrow} \mathcal{U}_C$$

Proof. The map ξ_C^r is clearly surjective on vertices. Moreover, given a morphism $U \to K$ in \mathcal{E}_C , choose a marked morphism $U \to U^r$ to a rigid chain in \mathcal{E}_C . Then for every $(a,b) \in U^r \setminus U$, the object $K \cup \{(a,b)\}$ will lie in \mathcal{E}_C over the same element of \mathcal{U}_C as K. We thus obtain a morphism

 $U^r \to \hat{K}$ lying over the original morphism in \mathcal{U}_C , showing that ξ_C^r is surjective on morphisms, and thus on higher simplices.

Moreover ξ_C^r detects and preserves marked edges and thin simplices. It will therefore suffice to show that ξ_C^r is injective. Let $K_i \in \mathcal{E}_C^r$ for i=1,2 such that $\xi_C^r(K_1) = \xi_C^r(K_2)$. Let us denote $K_i = \{(a_j^i,b_j^i)\}_{j=0}^{\ell_i}$ for i=1,2. Without loss of generality let us assume that we have some (a_s^1,b_s^1) such that this pair is not an element in K_2 . However, note that since $K_i \subset C$ for i=1,2 then there exists a map $K_2 \to \hat{K}_2$ where \hat{K}_2 is obtained from K_2 by appending the element (a_s^1,b_s^1) . By construction it follows that $\xi_C^r(K_2) = \xi_C^r(\hat{K}_2)$ since K_2 is rigid it follows that $\hat{K}_2 = K_2$ and therefore $K_1 = K_2$.

Lemma 3.24. Let $C \in \mathbb{E}^{n,k}_{(0,0)}$. Then the induced morphism

$$\xi_C \colon \mathcal{E}_C \stackrel{\simeq}{\longrightarrow} \mathcal{U}_C$$

is an equivalence of marked scaled simplicial sets.

Proof. Let $\iota: \mathcal{E}_C^r \to \mathcal{E}_C$ denote the obvious inclusion. Using Lemma 3.23 we can construct a map $s_C = \iota \circ (\xi_C^r)^{-1}$. It is clear that $\xi_C \circ s_C = \mathrm{id}$. Given K, observe that by construction $s_C \circ \xi_C(K)$ is rigid. Let $K \to K^r$ be a marked edge where K^r is rigid. Since the restriction of ξ_C to rigid objects is injective it follows that $s_C \circ \xi_C(K) = K^r$. This yields a marked homotopy from $s_C \circ \xi_C$ to the identity and the result follows.

Lemma 3.25. Let $C_i \in \mathbb{E}^{n,k}_{(0,0)}$ for i = 1, 2. Then there exists a chain K such that the intersection $\mathcal{E}_{C_1} \cap \mathcal{E}_{C_2} = \mathcal{E}_K$.

Proof. Immediate. \Box

Proposition 3.26. Let n, k two non-negative integers and consider $i \in [n]$ and $j \in [k]$. Then the morphism

 $\xi_{(i,j)}^{n,k} \colon \mathbb{E}_{(i,j)}^{n,k} \longrightarrow \mathbb{S}_{(i,j)}^{n,k}$

is an equivalence of marked scaled simplicial sets.

Proof. First let us observe that the map $\xi_{(i,j)}^{n,k}$ is an isomorphism if either n or k is equal to 0. Using an inductive argument it will suffice to show that the map $\xi_{(0,0)}^{n,k}$ is an equivalence. Note that we can cover $\mathbb{E}_{(i,j)}^{n,k}$ with the subsimplicial sets \mathcal{E}_C where C is a chain of maximal length. Since $\xi_{(0,0)}^{n,k}$ is surjective is covered by the subsimplicial sets \mathcal{U}_C . Applying [3, Lemma 3.2.13], we express $\mathbb{E}_{(0,0)}^{n,k}$ and $\mathbb{S}_{(0,0)}^{n,k}$ as the colimit over the same diagram of two homotopy cofibrant diagrams. We can now identify $\xi_{(0,0)}^{n,k}$ as the map induced by the natural transformation whose components are ξ_C . Therefore using Lemma 3.24 it follows that $\xi_{(0,0)}^{n,k}$ is a weak equivalence.

Definition 3.27. Let $\mathbb{O}^{n,k} = \mathfrak{C}[\Delta^n \times \Delta^k]$. We define a marking on $\mathbb{O}^{n,k}((i,j),(a,b))$ by declaring an edge marked if an only if its image in $\mathbb{O}^n(i,a) \times \mathbb{O}^k(j,b)$ is degenerate. If a,b=n,k we set the notation $\mathbb{O}^{n,k}((i,j),(a,b)) = \mathbb{O}^{n,k}_{(i,j)}$.

Lemma 3.28. The canonical morphism $p: \mathbb{O}^{n,k}((i,j),(a,b)) \to \mathbb{O}^n(i,a) \times \mathbb{O}^k(j,b)$ is a weak equivalence of marked simplicial sets.

Proof. The argument here is virtually identical to that given in Lemma 3.23, Lemma 3.24, and Proposition 3.26.

Definition 3.29. Let $\sigma: K_0 \subset K_1 \cdots \subset K_\ell$ be a simplex in $\mathbb{O}^{n,k}((i,j),(a,b))$ such that $K_0 \neq (i,j)$. Given $(x,y) \in K_0$ then it follows that σ is in the image of the map

$$\gamma_{x,y} \colon \mathbb{O}^{n,k}((i,j),(x,y)) \times \mathbb{O}^{n,k}((x,y),(a,b)) \longrightarrow \mathbb{O}^{n,k}((i,j),(a,b)).$$

Given a pair of simplices σ_1, σ_2 as above, let (A_i, B_i) denote the preimages of γ_i for i = 1, 2 under $\gamma_{x,y}$. We define $\mathcal{O}^{n,k}((i,j),(a,b))$ as a quotient of $\mathbb{O}^{n,k}((i,j),(a,b))$ by identifying those simplices σ_1, σ_2 as above such that their corresponding A_i 's get identified in $\mathbb{O}^n(i,x) \times \mathbb{O}^k(j,y)$ and $B_1 = B_2$.

Remark 3.30. Observe that the previous definition yields a factorization

$$\mathbb{O}^{n,k}((i,j),(a,b)) \xrightarrow{\alpha} \mathcal{O}^{n,k}((i,j),(a,b)) \xrightarrow{\beta} \mathbb{O}^{n}(i,a) \times \mathbb{O}^{k}(j,b)$$

Lemma 3.31. The morphisms in Remark 3.30 are equivalences of marked simplicial sets.

Proof. By Lemma 3.28, it suffices to show that α is an equivalence. For (i,j) < (x,y), let the distance from (x,y) to (i,j) be the maximal length of a chain in $\mathbb{O}^{n,k}((i,j),(x,y))$, using the convention that we count neither (i,j) nor (x,y) towards this length.

It is clear that if the distance from (a,b) to (i,j) is 0, α is an isomorphism. We then proceed by induction. Suppose that that statement is true for all (i,j) and (a,b) with distance less than r, and let (i,j) and (a,b) be distance r apart. We define a sequence of marked simplicial sets by setting

$$X_0 = \mathbb{O}^{n,k}((i,j),(a,b))$$

and then defining

$$\coprod_{\substack{d((i,j),(x,y))=\ell\\ \\ \coprod\\ d((i,j),(x,y))=\ell}} \mathbb{O}^{n,k}((i,j),(x,y)) \times \mathbb{O}^{n,k}((x,y),(a,b)) \longrightarrow X_{\ell}$$

We then note two facts:

• For (i,j) < (x,y) distance 0 apart, the canonical map

$$\mathbb{O}^{n,k}((i,j),(x,y))\times\mathbb{O}^{n,k}((x,y),(a,b))\to X_0$$

descends through an isomorphism to a map $\mathcal{O}^{n,k}((i,j),(x,y)) \times \mathbb{O}^{n,k}((x,y),(a,b)) \to X_0$

• For (i,j) < (x,y) distance ℓ apart, the canonical map

$$\mathbb{O}^{n,k}((i,j),(x,y))\times \mathbb{O}^{n,k}((x,y),(a,b))\to X_\ell$$

descends to a map $\mathcal{O}^{n,k}((i,j),(x,y)) \times \mathbb{O}^{n,k}((x,y),(a,b)) \to X_{\ell}$, since we have already quotiented out by the relations involving intermediate elements of lesser distance.

We can thus replace the pushout above with the pushout

$$\coprod_{\substack{d((i,j),(x,y))=\ell\\ \\ \coprod\\ d((i,j),(x,y))=\ell}} \mathcal{O}^{n,k}((i,j),(x,y)) \times \mathbb{O}^{n,k}((x,y),(a,b)) \longleftrightarrow X_{\ell}$$

where the upper horizontal map is now a cofibration. This means that, by our inductive hypothesis and Lemma 3.28, $X_{\ell} \to X_{\ell+1}$ is an pushout of an equivalence along a cofibration, and thus an equivalence of marked simplicial sets.

Since any intermediate element (i, j) < (x, y) < (a, b) must have distance from (i, j) strictly less than r, we see that $X_r = \mathcal{O}^{n,k}((i, j), (a, b))$. Thus, the composite map

$$\alpha \colon \mathbb{O}^{n,k}((i,j),(a,b)) = X_0 \longrightarrow X_r = \mathcal{O}^{n,k}((i,j),(a,b))$$

is an equivalence, as desired.

Proposition 3.32. Let $n, k \ge 0$ then the morphism $\pi_{(i,j)}^{n,k} : \mathbb{E}_{(i,j)}^{n,k} \to \mathbb{P}_{(i,j)}^{n,k}$ is a weak equivalence of marked scaled simplicial sets.

Proof. Since $\pi_{(i,j)}^{n,k}$ is an isomorphism whenever either n or k is equal to 0 it follows by an inductive argument that it will suffice to show that $\pi_{(0,0)}^{n,k}$ is an equivalence. We define a sequence of marked scaled simplicial sets beggining with

$$Y_0 = \mathbb{E}^{n,k}_{(0,0)}$$

Then we define

$$\coprod_{\substack{d((0,0),(x,y))=\ell\\ \\ \coprod\\ d((i,j),(x,y))=\ell}} \mathcal{O}^{n,k}((0,0),(x,y)) \times \mathbb{E}^{n,k}_{(x,y)} \longrightarrow Y_{\ell}$$

and observe that the top horizontal morphism is a cofibration. Additionally one sees that the left-most vertical morphism is an equivalence due to Lemma 3.31. It follows by construction that $Y_{n+k} = \mathbb{P}^{n,k}_{(0,0)}$ and since each $Y_{\ell} \to Y_{\ell+1}$ is a weak equivalence the result now follows.

Corollary 3.33. Let n, k two non-negative integers and consider $i \in [n]$ and $j \in [k]$. Then the morphism

$$\varepsilon_{(i,j)}^{n,k} \colon \mathbb{P}_{(i,j)}^{n,k} \longrightarrow \mathbb{S}_{(i,j)}^{n,k}$$

is an equivalence of marked scaled simplicial sets.

Proof of Theorem 3.17. As in [21, 3.2.1.13], it suffices to check this in the special case when $X_A \to A$ and $X_B \to B$ are identity morphisms on underlying simplicial sets, and both A and B are one of the following cases

- The scaled 2-simplex Δ^2_{\sharp} .
- The unscaled *n*-simplex Δ_{b}^{n} .

In the case where $A = \Delta_{\flat}^n$ and $B = \Delta_{\flat}^k$, the morphism

$$\varepsilon_X \colon \mathbb{S}t_{\phi}(\Delta^n \times \Delta^k)(i,j) \longrightarrow \left(\mathbb{S}t_{\Delta^n}(\Delta^n) \boxtimes \mathbb{S}t_{\Delta^k}(\Delta^k)\right)(i,j)$$

is precisely the morphism

$$\varepsilon_{(i,j)}^{n,k} \colon \mathbb{P}_{(i,j)}^{n,k} \longrightarrow \mathbb{S}_{(i,j)}^{n,k}$$

and thus is an equivalence of marked-scaled simplicial sets by Corollary 3.33. Each other case is a pushout of some $\varepsilon_{(i,j)}^{n,k}$ by a cofibration, and thus is also an equivalence.

3.3. Straightening and anodyne morphisms

This section serves as a stepping-stone to see that the bicategorical straightening is a left Quillen functor. in particular, we will show that St_S preserves MB-anodyne morphisms for any $S \in \operatorname{Set}_{\Delta}^{\operatorname{sc}}$.

Definition 3.34. Consider Λ_i^n for $0 \le i \le n$. For every $0 \le s \le n$ we define $\Lambda \mathcal{L}_i^n(s)$ to be the scaled subsimplicial set of $\mathcal{L}_{\flat}^n(s)$ consisting of those simplices $\sigma: S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n$ satisfying at least one of the following conditions:

- There exists $k \in [n]$ with $k \neq i$ such that, for every $j \in [n]$, $k \notin S_j$.
- There exists some $0 < j \le n$ such that $j \in S_0$ and there exists $0 \le \ell < j$ such that $\ell \ne i$.

Given Δ_T^n as in Definition 3.13 we define $(\Lambda \mathcal{L}_i^n)_T(s)$ using the inherited scaling from $\mathcal{L}_T^n(s)$.

Definition 3.35. Given a **MB** simplicial set of the form $\Delta_T^n := (\Delta^n, \flat, \flat \subset T)$ for some T, we denote by $(\Lambda_i^n)_T$ the horn with the induced marking and biscaling. We write $\operatorname{St}_{\Delta_{\flat}^n}(\Lambda_i^n)_T$ for the functor associated to the object $(\Lambda_i^n)_T \to \Delta_{\flat}^n$.

Remark 3.36. In some specific instances we will have $\Delta_T^n := (\Delta^n, \flat, \flat, \flat \subset T)$ where $T = \Delta^{\{i,j,k\}}$ a chosen 2-simplex in Δ^n . In that situation we will chose a subscript notation $\Delta_{\dagger}^n = (\Delta^n, \flat, \flat, \flat \subset \Delta^{\{i,j,k\}})$. This convention will also applied to previously defined constructions like for example $(\Lambda_i^n)_{\dagger}$ or $\mathbb{S}_{\Delta^n}(\Delta^n)_{\dagger}$.

Lemma 3.37. Let $\Delta_T^n = (\Delta^n, \flat, \flat \subseteq T)$. Then for every $0 \leqslant s \leqslant n$ the canonical morphism

$$\operatorname{St}_{\Delta_{\mathfrak{b}}^{n}}(\Lambda_{i}^{n})_{T}(s) \xrightarrow{\simeq} (\Lambda \mathcal{L}_{i}^{n})_{T}(s)$$

is MS-anodyne.

Proof. It is clear that for every $0 < s \le n$ we can pick the morphism to be an isomorphims on the underlying simplicial sets. We further note that the proof of Lemma 3.14 still holds in this setting. Consequently, the claim follows.

Definition 3.38. Let $n \ge 0$ and $0 \le s \le n$. We say that a (non-degenerate) simplex σ in $\mathcal{L}^n(s)$ is a *path* if it is of maximal dimension. Let \mathcal{P}^n_s be the set of such paths. We will define an total order on \mathcal{P}^n_s as follows:

Given a path $\sigma: S_0 \subset S_1 \subset S_2 \subset \cdots S_\ell$ one sees that $S_{i+1} \setminus S_i = \{a_{i+1}\}$ consists precisely in one element. Therefore we can identify σ with a list of elements

$$S_{\sigma} = \{a_i\}_{i=1}^{\ell}.$$

Note that by the maximality of σ , $S_0 = \{s\}$.

Suppose we are given two such lists $S_{\sigma} = \{a_i\}_{i=1}^{\ell}$ and $S_{\theta} = \{b_i\}_{i=1}^{\ell}$. We declare $\sigma < \theta$ if for the first index j for which $a_j \neq b_j$ then we have $a_j < b_j$.

Lemma 3.39. Let $\Delta^n_{\diamond} = (\Delta^n, \flat, \flat \subset \Delta^{\{0,1,n\}})$ and consider the induced morphism $(\Lambda \mathcal{L}^n_0)_{\diamond}(0) \to \mathcal{L}^n_{\diamond}(0)$. Collapsing the morphism $0 \to 01$ to a degenerate edge on both sides yields a map of scaled simplicial sets

$$\Lambda \mathcal{R}_0^n \longrightarrow \mathcal{R}^n$$

which is scaled anodyne.

Proof. We use the order from Definition 3.38 to add simplices to $\Lambda \mathcal{R}_0^n$. We will add simplices in reverse order, i.e. for any path σ , we denote by $X^{\geqslant \sigma}$ the scaled simplicial subset of \mathcal{R}^n obtained by adding to $\Lambda \mathcal{R}_0^n$ all paths θ such that $\theta \geqslant \sigma$.

The procedure yields a filtration

$$\Lambda \mathcal{R}_0^n = X^{\geqslant \sigma_0} \longrightarrow X^{\geqslant \sigma_1} \longrightarrow X^{\geqslant \sigma_2} \longrightarrow \cdots \longrightarrow \mathcal{R}^n$$

where we have labeled our paths σ_i so that $\sigma_i > \sigma_{i+1}$. The proof proceeds by showing that $X^{\geqslant \sigma_{i-1}} \to X^{\geqslant \sigma_i}$ is scaled anodyne for any i.

The proof proceeds by cases. We fix the notation that $S_{\sigma_i} = \{a_k\}_{k=1}^n$.

1. Suppose that $a_1 \neq 1$. We prove this case by showing that the top horizontal map in the pullback diagram

$$\begin{array}{ccc} A_{\sigma_i} & \stackrel{\alpha_i}{\longrightarrow} \Delta^n \\ \downarrow & & \downarrow^{\sigma_i} \\ X^{\geqslant \sigma_{i-1}} & \stackrel{X \geqslant \sigma_i}{\longrightarrow} X^{\geqslant \sigma_i} \end{array}$$

is itself scaled anodyne.

We see that A_{σ_i} is the union of the following faces of σ_i :

- The face $d_0(\sigma_i)$, since we will have $0 \le 1 < a_1$ in each S_k .
- The face $d_n(\sigma_i)$, since this face will always be missing $a_n \neq 0$.
- The face $d_j(\sigma_i)$ for every j such that $a_{j+1} > a_j$. This is because this will, equivalently, be the j^{th} face of the (greater) simplex with vertex list

$$\{a_1,\ldots,a_{j-1},a_{j+1},a_j,a_{j+2},\ldots,a_n\}.$$

To see that the inclusion of $A_{\sigma_i} \to \Delta^n$ is scaled anodyne, we first note that, by necessity, there is at least one face of Δ^n not contained in A_{σ_i} . We then choose $t \in [n]$ such that $d_t(\Delta^n)$ is not contained in A_{σ_i} .

If we let $j \in [n]$ be the an element such that j < t and $d_j(\Delta^n) \subset A_{\sigma_i}$. Similarly, let $\ell \in [n]$ be the smallest element such that $\ell > t$ and $d_\ell(\Delta^n) \subset A_{\sigma_i}$. A similar argument to [5, Lemma 1.10] (or Lemma 2.35) shows that it will suffice to see that the simplex $\Delta^{\{j,t,\ell\}}$ is scaled for every such j, t, and ℓ . It is easy to see that $\max(S_\ell) = \max(S_t)$. Consequently, we see that α_i is scaled anodyne, as desired.

2. Now suppose that $a_1 = 1$, and $S_{\sigma_i} \neq \{1, 2, \dots, n-1, n\}$. We now must instead consider the pullback diagram

as above, we see that B_{σ_i} consists of the faces

- $d_n(\sigma_i)$
- $d_j(\sigma_i)$ for each j such that $a_j < a_{j+1}$.

Since $S_{\sigma_i} \neq \{1, 2, \dots, n-1, n\}$, there exists some 1 < t < n such that B_{σ_i} does not contain the t^{th} face of $\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0$.

We can then consider the pullback diagram

$$C_{\sigma_i} \xrightarrow{\gamma_i} \Delta^{n-1}$$

$$\downarrow \qquad \qquad \downarrow^{d_0}$$

$$B_{\sigma_i} \longrightarrow \Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

an apply precisely the argument from the first case to $t \in [n-1]$ described above to find that γ_i is scaled anodyne. This means that, via a pushout, we may assume that B_{σ_i} contains the 0th face of $\Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0$. We can then repeat essentially the same argument, and thereby see that β_i is scaled anodyne.

3. If $S_{\sigma_i} = \{1, 2, \dots, n-1, n\}$, then the map $X^{\geqslant \sigma_{i-1}} \longrightarrow X^{\geqslant \sigma_i} = \mathbb{R}^n$ is an inclusion

$$\Lambda_0^n \coprod_{\Delta^{\{0,1\}}} \Delta^0 \longrightarrow \Delta^n \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

where $\Delta^{\{0,1,n\}}$ is scaled.

Lemma 3.40. Let $\Delta_{\dagger}^n = (\Delta^n, \flat, \flat \subset \Delta^{\{0,n-1,n\}})$ and consider the induced morphism $(\Lambda \mathcal{L}_n^n)_{\dagger}(0) \to \mathcal{L}_{\dagger}^n(0)$. Denote by \mathcal{T}^n (resp. $\Lambda \mathcal{T}_n^n$) the marked scaled simplicial set obtained from $\mathcal{L}_n^n(0)$ (resp. $(\Lambda \mathcal{L}_n^n)_{\dagger}(0)$) by marking the edges associated to the edge $(n-1) \longrightarrow n$ in Δ^n . Then the associated map

$$\Lambda \mathcal{T}_n^n \longrightarrow \mathcal{T}^n$$

is MS-anodyne.

Proof. The argument is nearly identical to the proof of Lemma 3.39. Using the same order as in that proof, we produce a filtration

$$\Lambda \mathcal{T}_n^n = X^{\geqslant \sigma_0} \longrightarrow X^{\geqslant \sigma_1} \longrightarrow \cdots \longrightarrow \mathcal{T}^n$$

and show each step is scaled anodyne.

As before, we set $S_{\sigma_i} = \{a_i\}_{i=1^n}$, and consider the pullback diagram

$$\begin{array}{ccc} A_{\sigma_i} & \stackrel{\alpha_i}{\longrightarrow} \Delta^n \\ \downarrow & & \downarrow^{\sigma_i} \\ X^{\geqslant \sigma_{i-1}} & \stackrel{X \geqslant \sigma_i}{\longrightarrow} X^{\geqslant \sigma_i} \end{array}$$

The case distinction now rests on whether or not $d_n\sigma_i$ factors through A_{σ_i} . The case when it does is formally identical to case (1) from Lemma 3.39.

If $d_n(\sigma_i)$ does not factor through A_{σ_i} , then $a_n = n$. There are again two cases, based on whether $S_{\sigma_i} = \{1, 2, \ldots, n-1, n\}$. The case $S_{\sigma_i} \neq \{1, 2, \ldots, n-1, n\}$ is identical to the corresponding case in Lemma 3.39. In the case $S_{\sigma_i} = \{1, 2, \ldots, n-1, n\}$ we find that $A_{\sigma_i} = \Lambda_n^n$, the last edge is marked, and $\Delta^{\{0,n-1,n\}}$ is scaled. This is a morphism of type (MS4), and thus is MS-anodyne.

Lemma 3.41. Let $\Delta_{*_i}^n = (\Delta^n, \flat, \flat \subset \Delta^{\{i-1,i,i+1\}})$ and consider the induced morphism $(\Lambda \mathcal{L}_i^n)_{*_i}(0) \to \mathcal{L}_{*_i}(0)$. Let \mathcal{S}^n (resp $\Lambda \mathcal{S}_i^n$) denote the marked scaled simplicial set obtained by marking the edges of the form $S \to S'$ such that $i, i+1 \in S$ but $i \notin S$ and such that $S' = S \cup \{i\}$. Then the induced morphism

$$\Lambda \mathcal{S}_i^n \longrightarrow \mathcal{S}_i^n$$

is MS-anodyne.

Proof. Let $S_{\tau} = \{1, 2, \dots, i-1, i+1, \dots, n-1, n, i\}$ and denote by $\hat{\tau}$ the smallest maximal simplex such that $\hat{\tau} > \tau$. We define a filtration as in Lemma 3.39 and Lemma 3.40 up until the stage $X^{\geqslant \hat{\tau}}$, yielding

$$\Lambda S_i^n \longrightarrow X^{\geqslant \sigma_1} \longrightarrow \cdots \longrightarrow X^{\geqslant \hat{\tau}} \longrightarrow S_i^n.$$

We will first prove that that each step of this factorization is MS-anodyne, making a distinction into 2 cases.

We consider the map $X^{\geqslant \sigma_{k-1}} \to X^{\sigma_k}$, and set $S_{\sigma_k} := \{a_j\}_{j=1}^n$. We again form the pullback

$$A_{\sigma_k} \xrightarrow{\alpha_k} \Delta^n \qquad \qquad \downarrow^{\sigma_k} \qquad \qquad \downarrow^{\sigma_k} \qquad \qquad X^{\geqslant \sigma_{k-1}} \xrightarrow{X \geqslant \sigma_k} X^{\geqslant \sigma_k}$$

We then have two cases.

- 1. If S_{σ_k} has as its last entry anything other than i, then A_{σ_k} consists of
 - The face $d_0(\sigma_k)$.
 - The face $d_n(\sigma_k)$.
 - The face $d_j(\sigma_k)$ for each j such that $a_{j+1} > j_j$.

The argument is then nearly identical to case (1) from Lemma 3.39.

- 2. If the last entry of S_{σ_k} is i, then A_{σ_k} consists of
 - The face $d_0(\sigma_k)$.
 - The face $d_i(\sigma_k)$ for each j such that $a_{i+1} > a_i$.

The remainder of the argument is nearly identical to case (2) of Lemma 3.39.

It now remains only for us to show that $X^{\geqslant \hat{\tau}} \to \mathcal{S}_i^n$ is **MS**-anodyne. For ease of notation, we set $Z := X^{\geqslant \hat{\tau}}$.

We now need to add the remaining simplices. Write Σ^{\leqslant} for the set of maximal simplices which are less than or equal to τ . Given $\theta \in \Sigma^{\leqslant}$, we write $S_{\theta} = \{b_j\}_{j=1}^n$ for the ordered vertex sequence, as usual. We further denote by \widehat{S}_{θ} the vertex sequence obtained by removing i. We will call a simplex $\theta \in \Sigma^{\leqslant}$ disordered if $\widehat{S}_{\theta} > \widehat{S}_{\tau}$. If $\widehat{S}_{\theta} = \widehat{S}_{\tau}$, we will call θ calm.³

Our first order of business is to add the disordered simplices in Σ^{\leq} . For each disordered θ , we define $Z^{\geqslant \theta}$ to be obtained from Z by adding all the disordered simplices σ for which $\sigma \geqslant \theta$. Applying the order induced on disordered simplices, we again obtain a filtration

$$Z \longrightarrow Z^{\geqslant \sigma_1} \longrightarrow Z^{\geqslant \sigma_2} \longrightarrow \cdots \longrightarrow Z^{\geqslant \gamma}$$

where γ is the minimal disordered simplex under the order <.

As before, we form a pullback diagram

$$B_{\sigma_k} \xrightarrow{\beta_k} \Delta^n \\ \downarrow \qquad \qquad \downarrow^{\sigma_k} \\ Z^{\geqslant \sigma_{k-1}} \longrightarrow Z^{\geqslant \sigma_k}$$

and show that β_k is MS-anodyne. Note that B_{σ_k} consists precisely of

- The face $d_0(\sigma_k)$.
- The face $d_n(\sigma_k)$ (since the final entry of S_{σ_k} cannot be i).
- The face $d_j(\sigma_k)$ for each j such that $a_{j+1} > a_j$.

The argument that β_k is anodyne is, by now, routine.

³If $\theta \in \Sigma^{\leq}$ is calm, then the entries of \widehat{S}_{θ} are in the linear order induced by the order on the integers. If θ is disordered, they are not.

We now turn to adding the calm simplices. Notice that τ is the maximal calm simplex. We now set $Y := Z^{\geqslant \gamma}$, and define a filtration

$$Y \longrightarrow Y^{\leqslant \sigma_1} \longrightarrow Y^{\leqslant \sigma_2} \longrightarrow \cdots \longrightarrow Y^{\leqslant \tau} = \mathbb{S}_i^n$$

By defining $Y^{\leq \theta}$ to be the union of Y with all of the calm simplices less than or equal to θ . For every calm σ_k other than τ itself, we obtain a pullback diagram

$$\begin{array}{ccc}
\Lambda_{\ell}^{n} & \xrightarrow{\eta_{k}} & \Delta^{n} \\
\downarrow & & \downarrow \sigma_{k} \\
Y^{\leqslant \sigma_{k-1}} & \longrightarrow & Y^{\leqslant \sigma_{k}}
\end{array}$$

where Λ^n_ℓ is an inner horn. If $S_{\sigma_k} = \{1, 2, 3, \dots, n-1, n\}$, then this is a Λ^n_i , and the scaling on $\Delta^n_{*_i}$ shows us that the simplex $\Delta^{\{i-1,i,i+1\}} \subset \sigma_k$ is scaled. On the other hand, if $S_{\sigma_k} \neq \{1, 2, 3, \dots, n-1, n\}$, the simplex $\Delta^{\{\ell-1,\ell,\ell+1\}} \subset \Lambda^n_\ell$ is already scaled in $\mathcal{L}^n_\flat(0) \subseteq \mathcal{L}^n_{*_i}(0)$. The morphism η_k is thus a scaled anodyne map, and the pushout is therefore **MS**-anodyne.

We are left only to add τ . However, in this case, we obtain a pullback diagram

$$\Lambda_n^n \xrightarrow{\mu} \Delta^n
\downarrow \qquad \qquad \downarrow^{\tau}
Y^{<\tau} \longrightarrow S_i^n$$

where the 2-simplex $\Delta^{\{0,n-1,n\}}$ is scaled and the edge $\Delta^{\{n-1,n\}}$ is marked. The result then follows from a pushout of type (MS4).

Proposition 3.42. Let S be a scaled simplicial set and let $\mathfrak{C}^{\mathrm{sc}}[S] \to \mathcal{C}$ be a functor of Set_{Δ}^+ -enriched categories. Consider an MB-anodyne morphism $i: A \to B$ in $(\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/S}$. Then for every $s \in S$ then induced map

$$\mathbb{S}t_{\phi}A(s) \longrightarrow \mathbb{S}t_{\phi}B(s)$$

is a trivial cofibration of marked scaled simplicial sets.

Proof. As in the proof of Proposition 3.16, we can assume that S = B, that ϕ is id: $\mathfrak{C}^{\mathrm{sc}}[S] \to \mathfrak{C}^{\mathrm{sc}}[S]$, and that i is one of the generators in Definition 2.18. We proceed to verify each case.

- A1) It is immediate that $\operatorname{St}_B A(s) \to \operatorname{St}_B B(s)$ is an isomorphism when $s \neq 0$. Lemma 3.41 shows that the map is MS-anodyne when s = 0.
- A2) Note that the morphism $\operatorname{St}_B A(s) \to \operatorname{St}_B B(s)$ is an isomorphism for $s \neq 0$. If s = 0 the map is an isomorphism on the underlying marked simplicial sets. Let $\hat{T} = T \cup \Delta^{\{0,1,4\}} \cup \Delta^{\{0,3,4\}}$ (see Definition 2.18) and let $\mathcal{L}_T^4(0)$ and $\mathcal{L}_{\hat{T}}^4$ be the simplicial sets defined in Definition 3.13 equipped with the marking given by the thin simplices in the base. We obtain a commutative diagram

$$\begin{array}{ccc} \mathbb{S} t_B A(0) & \longrightarrow & \mathbb{S} t_B B(0) \\ & & \downarrow \simeq & \downarrow \simeq \\ \mathcal{L}_T^4(0) & \longrightarrow & \mathcal{L}_{\hat{T}}^4(0) \end{array}$$

where the vertical morphisms are equivalences due to Lemma 3.14. We will show that the bottom morphism is an equivalence. Observe that once we manage to scale the simplices $0 \to 01 \to 014$ and $0 \to 03 \to 034$ then rest of the scaling follows using the argument given

in Lemma 3.14. We start by considering the 4-simplex given by

$$0 \rightarrow 01 \rightarrow 012 \rightarrow 0123 \rightarrow 01234$$

The only faces that are not scaled are precisely $\{0,01,01234\}$ and $\{0,0123,01234\}$. Consequently we can scale them using a pushout of type (MS2). Now we consider a 3-simplex

$$0 \to 01 \to 014 \to 01234$$

where all of its faces are now scaled except possibly the 3rd face. We further note that we can factor the last morphism as $014 \rightarrow 0134 \rightarrow 01234$ where both morphisms are marked. Therefore we can assume without loss of generality that the map $014 \rightarrow 01234$ is also marked. This allows us to scale the 3rd face using a pushout along a map of type (MS8). Inspecting the 3-simplex

$$0 \to 03 \to 0123 \to 01234$$

we see that we can add to the scaling $\{0,03,01234\}$. Finally let us consider

$$0 \to 03 \to 034 \to 01234$$
.

As we did before we factor the last map as a composite of marked morphisms $034 \rightarrow 0134 \rightarrow 01234$. The claim follows by a totally analogous argument as before.

- A3) Let * denote the vertex to which 0 and 1 get identified. Then it follows that the induced map $\mathbb{S}t_BA(s) \to \mathbb{S}t_BB(s)$ is an isomorphism for $s \neq *$. Lemma 3.39 shows that the map is MS-anodyne when s = 0.
- A4) It is immediate that $\mathbb{S}t_BA(s) \to \mathbb{S}t_BB(s)$ is an isomorphism for $s \neq 0$. Lemma 3.40 shows that the map is **MS**-anodyne when s = 0.
- S2) The induced map is an isomorphism for every object of Δ^2 .
- S3) The map is an isomorphism for every $s \in \Delta^3$ such that $s \neq 0$. We will prove the case i=1 leaving the case i=2 as an exercise to the reader. Let $\mathcal{L}^3_{U_1}(0)$ and $\mathcal{L}^3_{\sharp}(0)$ be as in Definition 3.13 and equip them with the marking induced by the thin simplex $\Delta^{\{0,1,2\}}$. We obtain a commutative diagram

$$\mathbb{S}t_B A(0) \longrightarrow \mathbb{S}t_B B(0)
\downarrow \simeq \qquad \qquad \downarrow \simeq
\mathcal{L}^3_{U_1}(0) \longrightarrow \mathcal{L}^3_{\sharp}(0)$$

where the vertical morphisms are equivalences due to Lemma 3.14. Therefore it will enough to show that the bottom morphism is an anodyne map of marked scaled simplicial sets. Consider the simplex $0 \to 01 \to 012 \to 0123$ and observe that all of its faces are scaled except the face missing 1. Therefore we can scale the 1-face using a pushout along an anodyne morphism as described in Lemma 2.41. Now we consider $0 \to 02 \to 012 \to 0123$ and we observe that we can scale the face missing 2 by another pushout. Finally we look at $0 \to 02 \to 023 \to 0123$ and we note that the last edge must be marked and that all of the faces are scaled except the face missing the vertex 3. Thefore another pushout along a morphism of type (MS8) yields the result.

- S4) & S5) The proof is very similar to the previous case and left to the reader.
- A5,S1 & E) Since these maps are always maximally thin scaled we can use Proposition 3.15 and apply

the pertinent proofs in Proposition 3.2.1.11 in [21].

3.4. Straightening over the point

In this section, we will prove two important results. We will show that the bicategorical straightening functor is left Quillen over any scaled simplicial set, and we will show that the straightening is an equivalence over the point. We fix the notation $St_{\Delta^0} = St_*$.

Definition 3.43. We define a an adjunction

$$L: \operatorname{Set}_{\Lambda}^{\mathbf{mb}} \iff \operatorname{Set}_{\Lambda}^{\mathbf{ms}}: R$$

where $L(X, E_X, T_X \subseteq C_X) := (X, E_X, C_X)$ and $R(Y, E_Y, T_Y) = (Y, E_Y, T_Y \subseteq T_Y)$. We note that $L \circ R = \text{id}$ and that the unit natural transformation $(X, E_X, T_X \subseteq C_X) \to (X, E_X, C_X \subseteq C_X)$ is **MB**-anodyne. It is easy to see that L preserves cofibrations and trivial cofibrations. In particular, we see that $L \dashv R$ is a Quillen equivalence.

Our goal in this section is to construct a natural transformation $St_* \Rightarrow L$ which is a levelwise weak equivalence. By general abstract nonsense, it will suffice to construct morphisms $\alpha_X : St_*(X) \rightarrow L(X)$ whenever X is one of the following generators

- $\Delta_{\flat}^n := (\Delta^n, \flat, \flat)$, for $n \ge 0$,
- $\Delta^2_{\dagger} := (\Delta^2, \flat, \flat \subset \Delta^2),$
- $\Delta^2_{\sharp} := (\Delta^2, \flat, \Delta^2),$
- $(\Delta^1)^{\sharp} := (\Delta^1, \Delta^1, \flat),$

and to prove that that the maps α_X are natural with respect to morphisms among generators. The next step is to give a precise description of the straightening functor applied to those generators.

Definition 3.44. Let $n \ge 0$ and define a simplicial set

$$Q^n := \bigsqcup_{0 \le i \le n} \mathbb{O}^{n+1}(i, n+1) /_{\sim}$$

where the relation \sim identifies simplices n-simplices $\sigma_1 \in \mathbb{O}^{n+1}(i, n+1)$ and $\sigma_2 \in \mathbb{O}^{n+1}(j, n+1)$ with $i \leq j$ whenever σ_1 is in the image of the map

$$\mathbb{O}^{n+1}(i,j) \times \Delta^n \xrightarrow{\mathrm{id} \times \sigma_2} \mathbb{O}^{n+1}(i,j) \times \mathbb{O}^{n+1}(j,n+1) \longrightarrow \mathbb{O}^{n+1}(i,n+1)$$

We further observe that the morphisms

$$\mathbb{O}^{n+1}(i,n+1) \longrightarrow \Delta^n, \ S \longmapsto \max \left(S \setminus \{n+1\} \right)$$

assemble into a map $\alpha_n: Q^n \to \Delta^n$. We wish now to upgrade Q^n to a scaled simplicial set. We do so by declaring a triangle $\sigma: \Delta^2 \to Q^n$ thin if and only if its image under p is degenerate in Δ^n . We denote this collection of thin triangles by T_{Q^n} .

Remark 3.45. Given an order preserving morphism $f:[n] \to [m]$ then it is straightforward to

check that we can produce a commutative diagram

$$Q^n \xrightarrow{Q(f)} Q^m$$

$$\downarrow^{\alpha_n} \qquad \downarrow^{\alpha_m}$$

$$\Delta^n \xrightarrow{f} \Delta^m$$

Lemma 3.46. We have the following isomorphisms of marked scaled simplicial sets

- $\operatorname{St}_*(\Delta^n_{\flat}) \simeq (Q^n, \flat, T_{Q^n}).$
- $\mathbb{S}_{\mathsf{t}_*}(\Delta^2_{\mathsf{t}}) = \mathbb{S}_{\mathsf{t}_*}(\Delta^2_{\mathsf{t}}) \simeq (Q^2, \flat, \sharp).$
- $\operatorname{St}_*((\Delta^1)^{\sharp}) = (Q^1, \sharp, \flat).$

Lemma 3.47. The morphism

$$\alpha_n \colon Q^n \longrightarrow \Delta_b^n$$

is a weak equivalence of marked scaled simplicial sets.

Proof. We construct a section $s: \Delta_{\flat}^n \to Q^n$ by sending $i \in [n]$ to the set $[0,i] \cup \{n+1\}$ and note that $\alpha_n \circ s = \mathrm{id}_{\Delta^n}$. To finish the proof we will construct a marked homotopy between id_{Q^n} and $s \circ \alpha_n$.

Let $\sigma: \Delta^k \to Q^n$ and pick a representative $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_k$ with $S_j \in \mathbb{O}^{n+1}(i, n+1)$ for $0 \le j \le k$. To ease the notation we will omit the element n+1 from the subsets S_j . Let us denote $s_j = \max(S_j)$ and observe that we can produce a diagram $H(-,\sigma): \Delta^1 \times \Delta^n \to Q^n$

It is straightforward to check that if $\sigma \sim \theta$ then $H(-,\sigma) = H(-,\theta)$. We have constructed now a natural transformation $\Delta^1 \times Q^n \to Q^n$. It is immediate to see that $H(0,-) = \mathrm{id}_{Q^n}$. In addition the fact that the bottom row in the diagram is equivalent to

$$[i, s_0] \longrightarrow [i, s_1] \longrightarrow \cdots \longrightarrow [i, s_{k-1}] \longrightarrow [i, s_k]$$

ensures that $H(1,-) = s \circ \alpha_n$. We conclude the proof by noting that the morphism $S_0 \to [i,s_0]$ gets collapsed to a degenerate edge and thus the homotopy is marked.

Proposition 3.48. There exists a natural transformation $\alpha : St_* \Rightarrow L$ which is a levelwise weak equivalence.

Proof. Using Lemma 3.46 is immediate to verify that the maps (together with decorated variants) $\alpha_n: Q^n \to \Delta^n$ assemble into a natural transformation $\alpha: \mathbb{S}\mathbf{t}_* \xrightarrow{L}$. To check that α is a levelwise equivalence, we note that due to the fact that both $\mathbb{S}\mathbf{t}_*$ and L are left adjoints which preserve cofibrations it will suffice to check on generators. We proceed case by case

- $\alpha_n: (Q^n, \flat, T_{Q^n}) \to (\Delta^n, \flat, \flat)$ is an equivalence due to Lemma 3.47.
- $\alpha_2^{\sharp}:(Q^2,\flat,\sharp)\to(\Delta^2,\flat,\sharp)$ is an equivalence since we can repeat the proof above with maximally scaled simplicial sets.
- $\alpha_1^{\sharp}: (Q^1, \sharp, \flat) \to (\Delta^1, \sharp, \flat)$ is an isomorphism.

Theorem 3.49. Let S be an scaled simplicial set, then the bicategorical straightening functor

$$\mathbb{S}t_S: (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S} \longrightarrow (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\operatorname{sc}}[S]^{\operatorname{op}}}$$

is a left Quillen functor.

Proof. Given a weak equivalence $f: X \longrightarrow Y$ in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}$, we can apply fibrant replacement to obtain a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \widetilde{X} \\ f \Big| & & \Big| \\ Y & \longrightarrow & \widetilde{Y} \end{array}$$

where the horizontal morphisms are MB-anodyne, and there vertical morphisms are weak equivalences.

Since $\mathbb{S}t_S$ preserves **MB**-andoyne morphisms, we may thus assume without loss of generality that X and Y are fibrant objects. By [4, Lemma 3.29], f then has a homotopy inverse g. Let

$$H: (\Delta^1)^{\sharp}_{\sharp} \times X \longrightarrow X$$

be a marked homotopy between $g \circ f$ and id_X over S. Then $St_S(H)$ factors as

$$\mathbb{S}t_{S}(X \times (\Delta^{1})^{\sharp}_{\sharp}) \stackrel{\varepsilon}{\longrightarrow} \mathbb{S}t_{S}(X) \boxtimes \mathbb{S}t_{*}((\Delta^{1})^{\sharp}_{\sharp}) \stackrel{\alpha}{\longrightarrow} \mathbb{S}t_{S}(X) \boxtimes (\Delta^{1})^{\sharp}_{\sharp} \longrightarrow \mathbb{S}t_{\Delta^{0}_{\flat}}(K^{\sharp})$$

Where ε is an equivalence by Theorem 3.17, α is an equivalence by Proposition 3.48, and the final map is an equivalence since $(\Delta^1)^{\sharp} \to \Delta^0$ is an equivalence of marked simplicial sets. Since $\operatorname{St}_S(g \circ f)$ and $\operatorname{St}_S(\operatorname{id}_X) = \operatorname{id}_{\operatorname{St}_S(X)}$ are both sections of $\operatorname{St}_S(H)$, they are thus equivalent in the homotopy category. An identical argument shows that $\operatorname{St}_S(f \circ g) \simeq \operatorname{id}_{\operatorname{St}_S(Y)}$, completing the proof. \square

Corollary 3.50. In particular the adjunction

$$\mathbb{S}t_*: \left(\operatorname{Set}_{\Delta}^{\mathbf{mb}}\right)_{/\Delta^0} \iff \operatorname{Set}_{\Delta}^{\mathbf{ms}}: \mathbb{U}n_*$$

is a Quillen equivalence.

Proof. By Proposition 3.48, St_* is naturally equivalent to a left Quillen equivalence. The corollary follows immediately.

3.5. Straightening over a simplex

As in [23, Ch. 2], the proof that our Grothendieck construction is a Quillen equivalence over a general scaled simplicial set will be bootstrapped from a proof over the n-simplices $(\Delta^n)_{\flat}$. In analogy to the method in op. cit., we will prove this case by constructing a mapping simplex for each 2-Cartesian fibration $X \to \Delta^n_{\flat}$ —a tractible **MB** simplicial set $\mathcal{M}_X \to \Delta^n_{\flat}$ which is equivalent to X over Δ^n_{\flat} . The majority of this section is given over to showing that we can decompose a 2-Cartesian fibration $X \to \Delta^n_{\flat}$ as a homotopy pushout of the restriction of X to Δ^{n-1} , which enables the inductive step of our proof.

Remark 3.51. The term "mapping simplex" used above is potentially misleading. in [21] and [23], a mapping simplex is a fibration over Δ^n explicitly constructed from a functor $\mathcal{F}: [n] \to \operatorname{Set}_{\Delta}^+$ or a $\mathcal{F}: \mathfrak{C}[\Delta^n] \to \operatorname{Set}_{\Delta}^+$. Our construction makes use of no such functor, and thus is not a true mapping simplex in this sense. The abuse of the term mapping simplex in the above exposition should be

⁴In contrast to the approach in [23, Ch. 2], we will not construct this 'mapping simplex' from an enriched functor, but rather as a pushout of **MB** simplicial sets over Δ_b^n .

seen as suggestive of the role this construction fills in our proof — one roughly analogous to the role of the mapping simplex in the proof of the $(\infty, 1)$ -categorical Grothendieck construction in [21, §3.2].

Definition 3.52. We define a marked biscaled simplicial set $(\Delta^n)^{\diamond} := (\Delta^n, E_{\diamond}^n, \flat \subset \sharp)$ where E_{\diamond}^n is the collection of all edges containing the vertex n. It is not hard to verify that the inclusion of the terminal vertex $\Delta^{\{n\}} \to (\Delta^n)^{\diamond}$ is **MB**-anodyne.

For the rest of this section, we fix be a 2-Cartesian fibration $p: X \to \Delta^n$ over the minimally scaled n-simplex. We consider the commutative diagram

$$X_n \times \Delta^{\{n\}} \xrightarrow{\alpha} X$$

$$\downarrow p$$

$$X_n \times (\Delta^n)^{\diamond} \longrightarrow \Delta^n$$

where X_n denotes the fibre over the vertex n and the dotted arrow exists due to the fact that the left vertical morphism is MB-anodyne.

Consider the inclusion morphism $\iota: \Delta^{[0,n-1]} \to \Delta^n$ and equip $\Delta^{[0,n-1]}$ with the structure of an **MB** simplicial set by declaring and edge (resp. triangle) marked (resp. thin, resp. lean) if its image in Δ^n is marked (resp. thin, resp. lean) in $(\Delta^n)^{\diamond}$. Notice that this amounts to equipping Δ^{n-1} with the minimal marking and thin-scaling, and the maximal lean-scaling.

We denote the restriction of X to Δ^{n-1} by $X|_{\Delta^{n-1}} := X \times_{\Delta^n} \Delta^{n-1}$, and denote the restriction of α to $X_n \times (\Delta^{n-1})^{\diamond}$ by α' . We construct an **MB** simplicial set \mathcal{M}_X over Δ^n by means of the pushout square

$$X_n \times (\Delta^{n-1})^{\diamond} \longrightarrow X_n \times (\Delta^n)^{\diamond}$$

$$\downarrow^{\alpha'} \qquad \qquad \downarrow$$

$$X|_{\Delta^{n-1}} \longrightarrow \mathcal{M}_X$$

Note that, since the top horizontal map is a cofibration, this is a homotopy pushout square in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\Delta_{\flat}^{n}}$. The morphism α and the inclusion $X|_{\Delta^{n-1}} \to X$ yield a cone over this diagram, and thus a canonical morphism $\omega : \mathcal{M}_{X} \to X$ over Δ^{n} . The key technical element in this section will be to show that ω is a weak equivalence in the 2-Cartesian model structure.

Definition 3.53. Let $\sigma: \Delta^k \to X$. Given $I \subset [k]$, we define

$$F_I(\sigma) = \{\theta : \Delta^I \to \mathcal{M}_X \mid \omega(\theta) = d_I(\sigma)\} \cup \{*\}.$$

Given $J \subset I \subseteq [k]$ and $\theta \in F_I(\sigma)$ such that $\theta \neq *$ we denote by $d_{J,I}(\theta)$ the image of θ in \mathcal{M}_X under the degeneracy operator induced by the inclusion $J \subset I$.

Definition 3.54. We define a **MB** simplicial set \mathcal{L}_X whose simplices $\sigma: \Delta^k \to \mathcal{L}_X$ are given by:

- A simplex $\hat{\sigma}: \Delta^k \to X$.
- For every non-empty subset $I \subseteq [k]$ an element $\theta_I \in F_I(\sigma)$. If $\theta_I = *$ we use the empty set notation $\theta_I = \emptyset$.

We impose to this data the following compatibility conditions

- H1) Given $J \subset I \subseteq [k]$ and $\theta_I \in F_I(\sigma)$ such that $\theta_I \neq \emptyset$ it follows that $d_{J,I}(\theta_I) = \theta_J$.
- H2) Given $I \subseteq [k]$ with $i_m = \max(I)$, then if $p \circ \hat{\sigma}(i_m) \neq n$ it follows that $\theta_I \neq \emptyset$.
- H3) Given $I \subseteq [k]$ such that for every $i \in I$ we have $p \circ \hat{\sigma}(i) = n$, then it follows that $\theta_I \neq \emptyset$.

Notice that by construction there is a canonical projection map $v: \mathcal{L}_X \to X$. We equip \mathcal{L}_X with the marking and biscaling induced by v.

Given a simplex $\sigma: \Delta^k \to \mathcal{L}_X$, we refer to the collection $\{\theta_I\}_{I \subset [k]}$ as the restriction data of σ .

Lemma 3.55. The projection map $v: \mathcal{L}_X \to X$ is a trivial fibration of MB simplicial sets.

Proof. Since v by definition detects all possible decorations, it will suffice to show that v is a trivial fibration on the underlying simplicial sets. Note that v is a bijection on 0-simplices. Given $k \ge 1$ we consider a lifting problem

$$\begin{array}{ccc}
\partial \Delta^k & \longrightarrow \mathcal{L}_X \\
\downarrow & & \downarrow \\
\Delta^k & \xrightarrow{\hat{\sigma}} & X
\end{array}$$

To produce the dotted arrow we use the bottom horizontal morphism as our choice for simplex in X. If $p \circ \hat{\sigma}(k) \neq n$ or $p \circ \hat{\sigma}$ is constant on n, we set $\theta_{[k]}$ to be the unique preimage of $\hat{\sigma}$ in \mathcal{M}_X . If $p \circ \hat{\sigma}(k) = n$ and it is not constant on n, we set $\theta_{[k]} = \emptyset$. The rest of the θ_I are always chosen according to top horizontal morphism. The compatibilities are clearly satisfied.

We construct a morphism $u: \mathcal{M}_X \to \mathcal{L}_X$ that sends a simplex $\theta: \Delta^k \to \mathcal{M}_X$ to the simplex $\omega(\theta)$ in X. For every $I \subseteq [k]$ we set $\theta_I = d_I(\theta)$. It is clear that u is a cofibration. It is not hard to see that u induces a bijection on the restriction to $\Delta^{[0,n-1]}$ and on the fibre over n.

Remark 3.56. Let $\pi = p \circ v : \mathcal{L}_X \to \Delta^n$. Given $\sigma : \Delta^k \to \mathcal{L}_X$ we fix the notation $\overline{\sigma} = \pi \circ \sigma$.

Definition 3.57. Let $\sigma: \Delta^k \to \mathcal{L}_X$ be a simplex such that $\overline{\sigma}(k) = n$. Let $\kappa_{\overline{\sigma}}$ be the first element in [k] such that $\overline{\sigma}(\kappa_{\overline{\sigma}}) = n$. We define a full subposet $Z_{\sigma} \subset [k] \times [n]$ consisting of

- Those vertices of the form $(x, \overline{\sigma}(x))$ with $x < \kappa_{\overline{\sigma}}$.
- Those vertices of the form (x, y) with $x \ge \kappa_{\overline{\sigma}}$ and $y \ge \overline{\sigma}(0)$.

We denote by \mathcal{Z}_{σ} the nerve $N(Z_{\sigma})$. Note that the projection $[k] \times [n] \to [n]$ yields a canonical map $\mathcal{Z}_{\sigma} \to \Delta^n$. We endow \mathcal{Z}_{σ} with the structure of an **MB** simplicial set by declaring an edge $(x_1, y_1) \to (x_2, y_2)$ marked if $x_1 = x_2 \geqslant \kappa_{\overline{\sigma}}$ and $y_2 = n$. A triangle is declared to be lean if the associated 2-simplex in Δ^k is degenerate. Finally we say that a triangle in \mathcal{Z}_{σ} is thin if it is already lean and its image in Δ^n is degenerate.

We call those non-degenerate simplices $\rho: \Delta^{\ell} \to \mathcal{Z}_{\sigma}$ which are not contained in any other non-degenerate simplex essential.

Remark 3.58. The avid reader might complain that our definition of \mathcal{Z}_{σ} only depends on $\overline{\sigma}$ and so should be denoted by $\mathcal{Z}_{\overline{\sigma}}$. The next definition will justify our notation.

Definition 3.59. Let $\sigma: \Delta^k \to \mathcal{L}_X$ such that $\overline{\sigma}(k) = n$. We define a subsimplicial set $\mathcal{X}_{\sigma} \subset \mathcal{Z}_{\sigma}$ (with the inherited marking and scalings) consisting of those simplices $\{(x_i, y_i)\}_{i=0}^{\ell}$ satisfying at least one of the conditions below

- i) We have $y_i = \overline{\sigma}(x_i)$ for $i = 0, \dots, \ell$.
- ii) There exists $I \subseteq [k]$ such that $\theta_I \neq \emptyset$ with $\overline{\sigma}(\max(I)) = n$ and $x_i \in I$ for $i = 0, \dots, \ell$.

Definition 3.60. Let $\sigma: \Delta^k \to \mathcal{L}_X$ such that $\overline{\sigma}(k) = n$ and suppose we are given a subset $I \subset [k]$ such that $\theta_I \neq \emptyset$ and such that $\overline{\sigma}(\max(I)) = n$. We construct a morphism

$$\Delta^I \times \Delta^{[\overline{\sigma}(0),n]} \longrightarrow (\Delta^n)^{\diamond} \times X_n$$

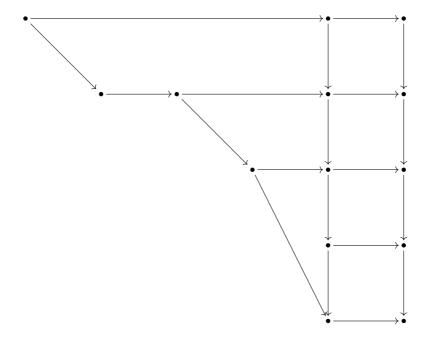


Figure 1: The poset Z_{σ} corresponding to the map [5] \rightarrow [4] given by the sequence of values 0, 1, 1, 2, 4, 4.

whose component at $(\Delta^n)^{\diamond}$ is given by $\Delta^I \times \Delta^{[\overline{\sigma}(0),n]} \to \Delta^{[\overline{\sigma}(0),n]} \to (\Delta^n)^{\diamond}$ and whose component at X_n is given by $\Delta^I \times \Delta^{[\overline{\sigma}(0),n]} \to \Delta^I \to X_n$ where the last morphism is induced from θ_I . We define a subposet $K_I \subset \Delta^I \times \Delta^{[\overline{\sigma}(0),n]}$ to be the intersection of $\Delta^I \times \Delta^{[\overline{\sigma}(0),n]}$ with \mathcal{Z}_{σ} . We

We define a subposet $K_I \subset \Delta^I \times \Delta^{[\sigma(0),n]}$ to be the intersection of $\Delta^I \times \Delta^{[\sigma(0),n]}$ with \mathcal{Z}_{σ} . We denote \mathcal{K}_I the **MB** simplicial set obtained by equipping K_I with the decorations induced from $(\Delta^n)^{\diamond} \times X_n$.

Remark 3.61. Observe that we can construct \mathcal{X}_{σ} as the union of Δ^k and every \mathcal{K}_I inside of \mathcal{Z}_{σ} .

Remark 3.62. Let $\sigma: \Delta^k \to \mathcal{L}_X$ such that $\overline{\sigma}(k) = n$. We define a morphism $\widetilde{f}_{\sigma}: \mathcal{X}_{\sigma} \to \mathcal{L}_X$ as follows:

- For simplices satisfying condition i) in Definition 3.59, \widetilde{f}_{σ} is simply σ .
- For simplices satisfying condition ii) in Definition 3.59, \widetilde{f}_{σ} is given by the composite

$$\mathcal{K}_I \longrightarrow \Delta^I \times \Delta^{[\overline{\sigma}(0),n]} \longrightarrow (\Delta^n)^{\diamond} \times X_n \stackrel{u}{\longrightarrow} \mathcal{L}_X.$$

One observes that this definition is compatible in the various intersections $\mathcal{K}_I \cap \mathcal{K}_J$ thus defining the desired morphism.

Definition 3.63. Let $\sigma: \Delta^k \to \mathcal{L}_X$ such that $\overline{\sigma}(k) = n$. We define a subsimplicial subset $\mathcal{X}_{\sigma}^{\uparrow} \to \mathcal{Z}_{\sigma}$ (with the induced decorations) consisting of those simplices $\{(x_i, y_i)\}_{i=0}^{\ell}$ that are either in \mathcal{X}_{σ} or satisfy the property:

• There exists $j \in [k]$ such that $x_i \neq j$ for every $i = 0, \dots, \ell$ and such that $\overline{\sigma}(d^j(k-1)) = n$.

Remark 3.64. Note that we can equivalently define $\mathcal{X}_{\sigma}^{\uparrow}$ to consist of those simplices that are either in \mathcal{X}_{σ} or that are contained in the image of

$$\mathcal{Z}_{d_i(\sigma)} \longrightarrow \mathcal{Z}_{\sigma}$$

where $d_i(\sigma)(k-1) = n$.

Definition 3.65. We define an order on the set of essential simplices of \mathcal{Z}_{σ} which we denote by " \prec ". Let ρ_i for i=1,2 be two essential simplices determined by the sequence of vertices $\{(x_j^i,y_j^i)\}_{j=0}^{r_i}$ for i=1,2. Let ε be the first index such that $\rho_1(\varepsilon) \neq \rho_2(\varepsilon)$. We say that $\rho_1 \prec \rho_2$ if precisely one the following conditions is satisfied:

- We have that $x_{\varepsilon}^1 = \kappa_{\sigma}$.
- We have $x_{\varepsilon}^{i} > \kappa_{\sigma}$ for i = 1, 2 and $y_{\varepsilon}^{1} > y_{\varepsilon}^{2}$.

Lemma 3.66. Let $\sigma: \Delta^k \to \mathcal{L}_X$ such that $\overline{\sigma}(k) = n$. Then the following morphisms are MB-anodyne:

$$\mathcal{X}_{\sigma} \longrightarrow \mathcal{X}_{\sigma}^{\uparrow} \longrightarrow \mathcal{Z}_{\sigma}.$$

Proof. If σ factors through \mathcal{M}_X , then $\mathcal{X}_{\sigma} = \mathcal{Z}_{\sigma}$, so we may assume without loss of generality that σ does not factor through \mathcal{M}_X .

We proceed by induction on k. Consider k=1, and note that $\mathcal{X}_{\sigma}=\mathcal{X}_{\sigma}^{\uparrow}$. Since σ does not factor through \mathcal{M}_{X} , we may assume $\sigma(0)\neq n$. There is thus some $\ell\geqslant 1$ such that the morphism $\mathcal{X}_{\sigma}\to\mathcal{Z}_{\sigma}$ can be identified with the inclusion

$$\psi_{\ell} \colon \Delta^1 \coprod_{\Delta^0} (\Delta^{\ell})^{\diamond} \longrightarrow (\Delta^{\ell+1})^{\dagger}$$

Where \dagger indicates marking where $i \to n$ is marked when $i \neq 0$, where every triangle is lean, and only those over degenerate triangles in Δ^n are thin. It follows immediately from Lemma 2.38 that this morphism is MB-anodyne.

We now suppose that the lemma holds in dimension k-1. By the inductive hypothesis,

$$\mathcal{X}_{\sigma} \longrightarrow \mathcal{X}_{\sigma}^{\uparrow}$$

is \mathbf{MB} -anodyne. Thus, it is sufficient for us to show that the morphism $\mathcal{X}_{\sigma}^{\uparrow} \longrightarrow \mathcal{Z}_{\sigma}$ is \mathbf{MB} -anodyne. We will add essential simplices of \mathcal{Z}_{σ} according to the order \prec . Given an essential simplex ρ , we denote by N_{ρ} the \mathbf{MB} simplicial subset of \mathcal{Z}_{σ} obtained by adding every essential simplex ρ' such that $\rho' \leq \rho$. We consider a pullback diagram

$$\begin{array}{ccc}
Q_{\rho} & \longrightarrow & \Delta^{r} \\
\downarrow & & \downarrow^{\rho} \\
N_{\rho'} & \longrightarrow & N_{\rho}
\end{array}$$

and we turn our attention to proving that the top horizontal morphism is **MB**-anodyne. Let us fix the notation $\rho = \{(x_j, y_j)\}_{j=0}^r$ and denote by θ the first index such that $x_{\theta} = \kappa_{\sigma}$.

We define three types of vertices $\varepsilon \in [r]$ of ρ .

Anterior vertices are those ε which have $x_{\varepsilon} < \kappa_{\sigma}$.

Recumbent vertices are those $\varepsilon \in [r]$ which have $x_{\varepsilon} > \kappa_{\sigma}$ and $x_{\varepsilon-1} < x_{\varepsilon}$. Note that this necessarily implies $y_{\varepsilon-1} = y_{\varepsilon}$

Plumb vertices are those $\varepsilon \in [r]$ which have $x_{\varepsilon} \geqslant \kappa_{\sigma}$ and $x_{\varepsilon-1} = x_{\varepsilon}$. Note that this necessarily implies that $y_{\varepsilon-1} < y_{\varepsilon}$. Note also that every ρ has at least one plumb vertex.

Notice that the only vertex which is not anterior, recumbent or plumb is θ . We call a vertex $\varepsilon \in [r]$ a downturn if ε is either recumbent or $\varepsilon = \theta$ and $x_{\varepsilon+1} = x_{\varepsilon}$. Note that ρ is uniquely determined by its set of downturns and the fact that it is essential.

We will prove three claims about these types of vertices, which then will enable us to complete the proof.

CLAIM 1: If ε is an anterior vertex, then $d_{\varepsilon}(\Delta^r) \subset Q_{\rho}$.

Subproof. Since ε is anterior, it is the only vertex of ρ whose first coordinate is x_{ε} . Consequently, $d_{\varepsilon}(\rho)$ factors through $\mathcal{Z}_{d_{\varepsilon}(\sigma)}$.

CLAIM 2: Let ε be a recumbent vertex. Then $d_{\varepsilon}(\Delta^r) \subset Q_{\rho}$.

Subproof. There are two cases. If $x_{\varepsilon+1} > x_{\varepsilon}$, then as before $d_{\varepsilon}(\rho)$ factors through $Z_{d_j(\sigma)}$ for some face operator d_j . If, on the other hand, $x_{\varepsilon+1} = x_{\varepsilon}$, then $d_{\varepsilon}(\rho)$ factors through a previous essential simplex.

CLAIM 3: Let X be a set of plumb vertices. Then $d_X(\Delta^r) \nsubseteq Q_{\rho}$.

Subproof. Since $d_X(\rho)$ contains a point with first coordinate j for every $j \in [k]$, we see that $d_X(\rho)$ cannot factor through $\mathcal{Z}_{d_j(\sigma)}$. Similarly, if $d_X(\rho)$ factors through \mathcal{K}_I for $I \subset [k]$, then I = [k], which would imply that σ factors through \mathcal{M}_X . Moreover, $d_X(\rho)$ cannot factor through σ , since it contains the vertex (x_{θ}, y_{θ}) .

Finally, if $\gamma \prec \rho$ is a previous simplex in our factorization, then $d_X(\rho)$ cannot factor through γ , since γ and ρ are uniquely determined by their sequences of downturns, and the only sequence of downturns containing $d_X(\rho)$ determine simplices greater than ρ under the order \prec .

To finish the proof we will consider two different cases. Each of this cases will be solved used inner-dull subsets (resp. right-dull) subsets. It is important to remark that since we can assume that $\dim(\sigma) > 1$ it follows that all the decorations in \mathcal{Z}_{σ} factor through $\mathcal{X}_{\sigma}^{\uparrow}$.

The first case is given precisely when the vertex r is recumbent. In this situation it follows that we can use Lemma 2.35 where the pivot point is given by the biggest plumb vertex. Since $r \neq \theta$ it follows that if r is not recumbent it must be plumb. In this cases the claim follow from Lemma 2.38.

Definition 3.67. Let $\sigma: \Delta^k \to \mathcal{L}_X$ such that $\overline{\sigma}(k) = n$ and let $\ell = n - \overline{\sigma}(\kappa_{\overline{\sigma}} - 1)$. For every morphism $f_{\sigma}: \mathcal{Z}_{\sigma} \to \mathcal{L}_X$ such that its restriction to \mathcal{X}_{σ} equals \widetilde{f}_{σ} as in Remark 3.62, we define a $(k + \ell)$ -simplex $\mathsf{B}(\sigma) \in \mathcal{L}_X$ to be the composite

$$\Delta^{k+\ell} \xrightarrow{\rho_{\sigma}} \mathcal{Z}_{\sigma} \xrightarrow{f_{\sigma}} \mathcal{L}_{X}$$

Where

$$\rho_{\sigma} \colon [k+\ell] \longrightarrow Z_{\sigma}$$

$$j \longmapsto \begin{cases} (j, \overline{\sigma}(j)) & 0 \leqslant j \leqslant \kappa_{\overline{\sigma}} - 1 \\ (\kappa_{\overline{\sigma}}, \overline{\sigma}(\kappa_{\overline{\sigma}} - 1)) & j = \kappa_{\overline{\sigma}} \\ (\kappa_{\overline{\sigma}}, j), & \kappa_{\overline{\sigma}} < j \leqslant \kappa_{\overline{\sigma}} + \ell \\ (j - \ell, n), & \kappa_{\overline{\sigma}} + \ell + 1 \leqslant j \leqslant k + \ell \end{cases}$$

Remark 3.68. We can equivalently characterise the simplex $B(\sigma)$ in Definition 3.67 as the smallest essential simplex of \mathcal{Z}_{σ} under the order \prec of Definition 3.65 that does not factor through $\mathcal{X}_{\sigma}^{\uparrow}$.

Remark 3.69. There are two key parameters which we will use to analyze the simplices $B(\sigma)$, for $\sigma: \Delta^k \to \mathcal{L}_X$. One is the fundamental vertex κ_{σ} — the first vertex such that $\overline{\sigma}(\kappa_{\sigma}) = n$. The other is the *terminal size* of σ : the number of vertices $j \in [k]$ such that $\overline{\sigma}(j) = n$. We will denote the terminal size of σ by

$$\nu_{\sigma} = |\{j \in [k] \mid \overline{\sigma}(j) = n\}|.$$

Notice that the terminal size of $B(\sigma)$ is always equal to the terminal size of σ .

To make use of the simplices $B(\sigma)$ in an inductive pushout argument, we will need to ensure we can make sufficiently compatible choices of maps

$$f_{\sigma} \colon \mathcal{Z}_{\sigma} \longrightarrow \mathcal{L}_{X}$$

to define our choices of $B(\sigma)$.

Proposition 3.70. There exists a collection indexed by the simplices of \mathcal{L}_X

$$\mathfrak{I} := \{ f_{\sigma} : \mathcal{Z}_{\sigma} \to \mathcal{L}_{X} \mid \sigma : \Delta^{k} \to \mathcal{L}_{X} \}.$$

With the following properties:

- i) The restriction of f_{σ} to \mathcal{X}_{σ} equals \tilde{f}_{σ} as in Remark 3.62.
- ii) Given a face operator $d_i: [k-1] \to [k]$ such that $\overline{\sigma}(d^j(k-1)) = n$ we have that the composite

$$\mathcal{Z}_{d_i(\sigma)} \longrightarrow \mathcal{Z}_{\sigma} \xrightarrow{f_{\sigma}} \mathcal{L}_X$$

equals $f_{d_i(\sigma)}$.

iii) If $\sigma \subseteq B(\tau)$, where $\tau \subsetneq \sigma$, then $B(\sigma)$ is degenerate on a simplex $\rho \subseteq B(\tau)$.

Remark 3.71. Of key importance to our argument is the fact that, if $\sigma \subseteq \mathsf{B}(\tau)$ and $\tau \subseteq \sigma$, then $\overline{\sigma}^{-1}(n), \overline{\tau}^{-1}(n)$, and $\overline{\mathsf{B}(\tau)}^{-1}(n)$ have the same cardinality, and σ, τ , and $\mathsf{B}(\sigma)$ agree on corresponding simplex.

Definition 3.72. Set $\Xi_k = \{\sigma : \Delta^r \to \mathcal{L}_X \mid r \leqslant k\}$. We will call a collection

$$\mathfrak{I} := \{ f_{\sigma} : \mathcal{Z}_{\sigma} \to \mathcal{L}_X \}_{\sigma \in \Xi_k}$$

a compatible k-collection if it satisfies conditions i), ii), and iii) above.

Lemma 3.73. Let \mathfrak{I}_{k-1} be a compatible (k-1)-collection, and let $\sigma: \Delta^k \to \mathcal{L}_X$ such that σ does not factor through \mathcal{M}_X . Suppose that there is a simplex $\tau: \Delta^s \to \mathcal{L}_X$ with s < k such that

- There is an inclusion $\sigma \subset \mathsf{B}(\tau)$.
- The terminal sizes agree, i.e. $\nu_{\sigma} = \nu_{\mathsf{B}(\tau)}$.

Then there is a subsimplex $\gamma \subset \tau$ such that

- 1. There is an inclusion $\gamma \subseteq \sigma$
- 2. There is an inclusion $\sigma \subset \mathsf{B}(\gamma)$
- 3. The terminal sizes ν_{σ} and $\nu_{\mathsf{B}(\gamma)}$ agree.

Proof. One need only restrict to the maximal sub-simplex $\tau \cap \sigma$ that factors through τ and σ both. Conditions (1) and (3) are immediate, and it is an easy check to see that $\sigma \subset B(\tau \cap \sigma)$.

Lemma 3.74. Let \mathfrak{I}_{k-1} be a compatible (k-1)-collection. Let $\sigma: \Delta^k \to \mathcal{L}_X$ and assume that σ does not factor through \mathcal{M}_X . Let us suppose there exists a pair $\tau_i: \Delta^{s_i} \to \mathcal{L}_X$ with $s_i < k$ for i=1,2; such that $\sigma \subseteq \mathsf{B}(\tau_i)$ and $\tau_i \subseteq \sigma$. Then $\tau_1 \subseteq \tau_2$ or $\tau_2 \subseteq \tau_1$.

Proof. We can partition Δ^{s_i} into two parts: $\tau_i^{-1}([0, n-1])$ and $\tau_i^{-1}(n)$. We identify each of these with subsets of Δ^k . By Remark 3.71, $\tau_1^{-1}(n) = \tau_2^{-1}(n) = \sigma^{-1}(n)$. Since σ does not factor through \mathcal{M}_X , the initial vertex of σ must factor through each τ_i . Since the only vertices j of $\mathsf{B}(\tau_i)$ which

are not vertices of τ_i satisfy with $j \ge \kappa_{\tau_i}$, we see that for $j \in [k]$ such that $j < \kappa_{\tau_i}$, $\sigma(j)$ must factor through τ_i .

Thus, we see that τ_i obtained from σ by deleting the vertices $j \in [k]$ such that $\kappa_{\tau_i} < j < \kappa_{\sigma}$. Thus, if $\kappa_{\tau_1} \leq \kappa_{\tau_2}$, then $\tau_1 \subseteq \tau_2$.

Corollary 3.75. Let \mathfrak{I}_{k-1} be a compatible (k-1)-collection, and let $\sigma:\Delta^k\to\mathcal{L}_X$ be a simplex that does not factor through \mathcal{L}_X . If there is any $\tau:\Delta^s\to\mathcal{L}_X$ with $\sigma\subseteq\mathsf{B}(\tau)$ and $\tau\subsetneq\sigma$, then there is a unique minimal such simplex. Moreover, if there is such a simplex τ with $\sigma=\mathsf{B}(\tau)$, then one such simplex is minimal.

Proof. The first statement is an immediate consequence of the previous lemma. To prove the second claim suppose that we have $\rho \subseteq \tau$ such that $\mathsf{B}(\tau) \subseteq \mathsf{B}(\rho)$. First we observe that $\nu_{\tau} = \nu_{\rho}$. Let μ_{ρ} be the biggest element of $\mathsf{B}(\rho)$ that does not lie over n and such that μ_{ρ} and $\mu_{\rho} - 1$ lie over the same vertex of Δ^n . We similarly define μ_{τ} as the biggest element of $\mathsf{B}(\rho)$ contained in $\mathsf{B}(\tau)$ satisfying the analogous property as before. Note that such elements always exist by construction. An easy argument then shows that $\mu_{\tau} = \mu_{\rho}$ and our claim follows from dimension counting.

Definition 3.76. Suppose given a compatible (k-1)-collection \mathcal{I}_{k-1} and $\sigma: \Delta^k \to \mathcal{L}_X$. If it exists, we call the minimal simplex of Corollary 3.75 the *capsule* of σ . We say that σ is *encapsulated* if it admits a capsule.

There is one final fact to establish: that there is a way of choosing a compatible degeneracy to ensure condition iii). Given $\sigma: \Delta^k \to \mathcal{L}_X$ which does not factor through \mathcal{M}_X , we denote by \mathcal{R}_{σ} the pullback

$$\mathcal{R}_{\sigma} \longrightarrow \Delta^{k+\ell}$$
 $\downarrow \qquad \qquad \downarrow^{
ho_{\sigma}}$
 $\mathcal{X}_{\sigma}^{\uparrow} \longrightarrow \mathcal{Z}_{\sigma}$

Given a compatible (k-1)-collection $\mathfrak{I}_{(k-1)}$, the compatibilities (1) and (2) allow us to define a map

$$\tilde{f}_{\sigma}^{\uparrow} \colon \mathcal{X}_{\sigma}^{\uparrow} \longrightarrow \mathcal{L}_{X}$$

for each $\sigma: \Delta^k \to \mathcal{L}_X$ which extends \tilde{f}_{σ} , and which agrees with $f_{d_j(\sigma)}$ for each face operator d_j such that $d_j(\sigma)(k-1) = n$.

Lemma 3.77. Let \mathfrak{I}_{k-1} be a compatible (k-1)-collection, and suppose that $\sigma: \Delta^k \to \mathcal{L}_X$ is encapsulated with capsule τ . Then for each $\zeta: \Delta^r \to \mathcal{R}_{\sigma}$ such that ζ hits both $\rho_{\sigma}(\kappa_{\sigma}-1)$ and $\rho_{\sigma}(\kappa_{\sigma})$, then $\tilde{f}_{\sigma}^{\uparrow}(\zeta)$ is degenerate on those vertices.

Proof. Note that our assumption means that ζ does not factor through σ .

First suppose that ζ factors through $\mathcal{Z}_{d_j(\sigma)}$ for $j \leqslant \kappa_{\tau} - 1$. Then we note that $d_j(\sigma) \subset \mathsf{B}(d_j(\tau))$, and $\nu_{d_j(\sigma)} = \nu_{\mathsf{B}(d_j(\tau))}$, so by Lemma 3.73 and the fact that \mathcal{I}_{k-1} satisfies iii') we see that $\tilde{f}_{\sigma}^{\uparrow}(\zeta)$ is degenerate. An identical argument holds when $j > \kappa_{\sigma}$.

If ζ factors through $\mathcal{Z}_{d_j(\sigma)}$ for $\kappa_{\tau} \leqslant j \leqslant \kappa_{\sigma}$, then $\sigma(j)$ is not in τ . Thus $\tau \subset d_j(\sigma)$, $d_j(\sigma) \subset \mathsf{B}(\tau)$, and so since \mathfrak{I}_{k-1} satisfies condition iii'), ζ is degenerate.

Finally, suppose that ζ factors through \mathcal{K}_I^{σ} for some $\theta_I^{\sigma} \neq \emptyset$. Then $I \cap [s] = J$ has $\theta_J^{\tau} \neq \emptyset$, and we can factor $\tilde{f}_{\sigma}^{\uparrow}(\zeta)$ through u as

$$\Delta^r \longrightarrow \Delta^I \times \Delta^{[\sigma(0),n]} \longrightarrow \Delta^n \times X_n$$

By construction, the first factor of this simplex is degenerate at the desired vertex. The second factor can be equivalently factored through $\Delta^J \times \Delta^{[\tau(0),n]}$ and thus is degenerate at the desired vertex as well. Thus $\tilde{f}_{\sigma}^{\uparrow}(\zeta)$ is degenerate.

Corollary 3.78. Let \mathfrak{I}_{k-1} be a compatible (k-1)-collection. Suppose that $\sigma: \Delta^k \to \mathcal{L}_X$ is encapsulated, and let τ be the capsule for σ . Then there is a subsimplex $\gamma \subset \mathsf{B}(\tau)$ and a degeneracy operator s_{α} such that the diagram

$$\mathcal{R}_{\sigma} \longrightarrow \Delta^{k+\ell}$$

$$\downarrow \qquad \qquad \downarrow s_{\alpha}(\gamma)$$
 $\mathcal{X}_{\sigma}^{\uparrow} \xrightarrow{\widetilde{f}_{\sigma}^{\uparrow}} \mathcal{L}_{X}$

commutes.

With this corollary in hand we can now return to Proposition 3.70.

Proof (of Proposition 3.70). First let us observe that the choices of f_{σ} for every $\sigma: \Delta^k \to \mathcal{M}_X \subset \mathcal{L}_X$ are already made since $\mathcal{X}_{\sigma} = \mathcal{Z}_{\sigma}$. It is also easy to check that the rest of the conditions hold for those choices. Therefore we can restrict our attention to producing the choices for simplices $\sigma: \Delta^k \to \mathcal{L}_X$ that do not factor through \mathcal{M}_X .

We will inductively compatible k-collections \mathcal{I}_k for every $k \geqslant 1$. Before commencing our argument we will make a preliminary definition. Given $\sigma: \Delta^k \to \mathcal{L}_X$ we define $\mathcal{Y}_{\sigma}^{\uparrow}$ to be the subsimplicial subset (with the inherited decorations) of \mathcal{Z}_{σ} whose simplices are those of $\mathcal{X}_{\sigma}^{\uparrow}$ in addition to the simplex $\mathsf{B}(\sigma)$. It follows from the argument given in Lemma 3.66 that the inclusion $\mathcal{Y}_{\sigma}^{\uparrow} \to \mathcal{Z}_{\sigma}$ is MB-anodyne.

For every $e: \Delta^1 \to \mathcal{L}_X$ we fix the choice of f_e which is guaranteed by Lemma 3.66. In this ground case, there are no conditions to check. Let us consider a triangle $\sigma: \Delta^2 \to \mathcal{L}_X$. Using the previous choices the can extend the map \widetilde{f}_{σ} to a morphism

$$f_{\sigma}^{\uparrow} \colon \mathcal{X}_{\sigma}^{\uparrow} \longrightarrow \mathcal{L}_{X}$$

We distinguish now two cases. Suppose that σ is not contained in some $\mathsf{B}(e)$ for $e:\Delta^1\to\mathcal{L}_X$. Then we define f_σ to be an extension of f_σ^\uparrow to \mathcal{Z}_σ . If $\sigma\subseteq\mathsf{B}(e)$ we can assume that $e\subset\sigma$ since otherwise we have $\sigma\in\mathcal{M}_X$. We extend f_σ^\uparrow to a map $\mathcal{Y}_\sigma^\uparrow\to\mathcal{L}_X$ by sending $\mathsf{B}(\sigma)$ to the following simplex: Let $\sigma_e:\Delta^r\to\mathcal{L}_X$ be the simplex obtained by forgetting every vertex j in $\mathsf{B}(e)$ such that $j\leqslant\kappa_{\overline{\sigma}}-1$ and such that is not in σ . We can now map $\mathsf{B}(\sigma)$ to $s_\alpha(\sigma_e)$ where $\alpha=\kappa_{\overline{\sigma}}-1$ and consequently condition iii is satisfied. This means that we can construct a compatible 1-collection \mathcal{I}_1 .

Now suppose we have a compatible (k-1)-collection \mathcal{I}_{k-1} . Let $\sigma: \Delta^k \to \mathcal{L}_X$ be a simplex. If σ is not encapsulated, then we may define f_{σ} by solving the lifting problem

$$\mathcal{X}_{\sigma}^{\uparrow} \xrightarrow{\hat{f}_{\sigma}^{\uparrow}} \mathcal{L}_{X}$$
 \downarrow
 \mathcal{Z}_{σ}

using 3.66. If σ is encapsulated with capsule τ , we can use Corollary 3.78 to define a map

$$\mathcal{Y}_{\sigma}^{\uparrow} \longrightarrow \mathcal{L}_{X}$$

which sends $B(\sigma)$ to the degenerate simplex described in Corollary 3.78. Solving the corresponding lifting problem yields an f_{σ} satisfying i), ii), and iii). Thus, we can extend \mathcal{I}_{k-1} to a compatible k-collection, as desired.

Proposition 3.79. The cofibration $u: \mathcal{M}_X \to \mathcal{L}_X$ is MB-anodyne.

Proof. We say that a simplex $\sigma: \Delta^k \to \mathcal{L}_X$ is wide if it is not contained in the image of u. Let

 $\sigma: \Delta^k \to \mathcal{L}_X$ and recall the definition $\nu_{\sigma} = |\{j \in [k] \mid \overline{\sigma}(j) = n\}|$. We produce a filtration

$$\mathcal{M}_X \to S^1 \to S^2 \to \cdots \to \mathcal{L}_X$$

where S^{ℓ} consists of those simplices σ in \mathcal{L}_X that either factor through \mathcal{M}_X or they satisfy $\nu_{\sigma} \leqslant \ell$. We will fix the convention $S^0 = \mathcal{M}_X$. We will show that each step in the filtration is **MB**-anodyne. Let us fix once and for all a choice of $f_{\sigma}: \mathcal{Z}_{\sigma} \to \mathcal{L}_X$ for every $\sigma: \Delta^k \to \mathcal{L}_X$ with the properties listed in Proposition 3.70. First, let us observe that given $\sigma: \Delta^k \to S^{\ell}$ it follows that the morphisms f_{σ} also factor through S^{ℓ} . We can now define $S^{(\ell,s)}$ to consist in those simplices contained in S^{ℓ} in addition to the simplices $B(\sigma)$ for $\sigma: \Delta^k \to \mathcal{L}_X$ wide and non-degenerate, such that $k \leqslant s$ and $\nu_{\sigma} = \ell + 1$. This produces a filtration

$$S^{\ell-1} \to S^{(\ell-1,\ell)} \to S^{(\ell-1,\ell+1)} \to \cdots \to S^{\ell}$$

We fix the convention $S^{\ell} = S^{(\ell,\ell)}$. Let us consider a pullback diagram

$$A_{\sigma} \longrightarrow \mathcal{Z}_{\sigma}$$

$$\downarrow \qquad \qquad \downarrow f_{\sigma}$$

$$S^{(\ell,s-1)} \longrightarrow S^{(\ell,s)}$$

where $\sigma: \Delta^s \to \mathcal{L}_X$ does not factor through $S^{(\ell,s-1)}$. Then it follows by construction that A_{σ} contains every simplex of \mathcal{Z}_{σ} except the simplex $\mathsf{B}(\sigma)$. To check that the top horizontal morphism is \mathbf{MB} -anodyne, it suffices to apply Lemma 2.35 with pivot point $\kappa_{\overline{\sigma}}$ after observing that the restriction of $\mathsf{B}(\sigma)$ to A_{σ} consists precisely in the union of the following $(s + \ell - 1)$ -dimensional faces:

- The face that misses the vertex j for $0 \le j \le \kappa_{\overline{\sigma}} 1$. This is because this simplex either factors through $\mathcal{Z}_{d_j(\sigma)}$ or it is contained in \mathcal{M}_X .
- The face that misses the vertex j for $\kappa_{\overline{\sigma}} + \ell \leq j \leq s + \ell$. This is because those faces have strictly smaller parameter $\nu_{d_j(\sigma)}$ if $\nu_{\sigma} > 1$ or they are already in \mathcal{M}_X if $\nu_{\sigma} = 1$.

To finish the proof we observe that given $\sigma: \Delta^s \to \mathcal{L}_X$ such that σ factors through $S^{(\ell,s-1)}$ but $\nu_{\sigma} = \ell + 1$ then it follows by condition iii) in Proposition 3.70 that $\mathsf{B}(\sigma)$ is already contained in $S^{(\ell,s-1)}$. This together with previous discussion implies that $S^{(\ell,s-1)} \to S^{(\ell,s)}$ is **MB**-anodyne. \square

We can distill the key upshot of the preceding technical arguments into a single, simple corollary.

Corollary 3.80. For any 2-Cartesian fibration $X \to \Delta_b^n$, the square

$$X_n \times (\Delta^{n-1})^{\diamond} \longrightarrow X_n \times (\Delta^n)^{\diamond}$$

$$\downarrow \qquad \qquad \downarrow$$

$$X|_{\Delta^{n-1}} \longrightarrow X$$

is homotopy pushout.

3.5.1. The equivalence over a simplex

Having now established the necessary preliminaries, we turn to the proof that the straightening is an equivalence over the minimally-scaled simplex. With few exceptions, the arguments from here on out are standard, and follow the general shape of the analogous arguments given in [21] and [23]. We begin with a lemma, which allows us to more easily apply the straightening to our homotopy pushout.

Lemma 3.81. Consider the inclusion $(\Delta^{n-1})^{\diamond} \to (\Delta^n)^{\diamond}$ as a morphism in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{\Delta_{\flat}^n}$. Then for every $0 \leq i < n$, the induced morphism

$$\psi \colon \mathbb{S}_{\mathsf{L}^n_{\mathsf{b}}}((\Delta^{n-1})^{\diamond})(i) \longrightarrow \mathbb{S}_{\mathsf{L}^n_{\mathsf{b}}}((\Delta^n)^{\diamond})(i)$$

is an equivalence of marked-scaled simplicial sets.

Proof. To begin, we examine the morphism on underlying marked simplicial sets. Consider the pushouts

$$(\Delta^{n-1})^{\diamond} \longrightarrow ((\Delta^{n-1})^{\diamond})^{\triangleright}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\Delta^{n} \longrightarrow X$$

and

$$(\Delta^n)^{\diamond} \longrightarrow ((\Delta^n)^{\diamond})^{\triangleright}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\Delta^n \longrightarrow Y$$

and the induced map

$$\phi \colon \mathfrak{C}^{\mathrm{sc}}[X](i,*) \longrightarrow \mathfrak{C}^{\mathrm{sc}}[Y](i,*)$$

We first note that, since i < n, we have that $\mathfrak{C}^{\mathrm{sc}}[X](i,*) = \mathfrak{C}[((\Delta^{n-1})^{\diamond})^{\triangleright}](i,*)$. From the definition, we then have that

$$\mathfrak{C}^{\mathrm{sc}}[X](i,*) \cong \mathrm{N}(\mathbb{P}(\{i+1,\ldots,n-1\}))^{\flat}$$

and

$$\mathfrak{C}^{\mathrm{sc}}[Y](i,*) \cong \mathrm{N}(\mathbb{P}(\{i+1,\ldots,n\}))^{\dagger}$$

where \dagger indicates the marking in which precisely the non-degenerate morphisms $S \to S \cup \{n\}$ are marked.

We note that, on underlying marked simplicial sets, this means that ϕ can be identified with the morphism

$$\mathfrak{C}^{\mathrm{sc}}[X](i,*)\times\{0\} \longrightarrow \mathfrak{C}^{\mathrm{sc}}[X](i,*)\times(\Delta^1)^{\sharp}.$$

We will show that this yields an equivalence of marked-scaled simplicial sets by showing that both scalings are equivalent to the maximal scaling.

We claim that the morphisms

$$f_n^i \colon \mathbb{S}t_{\Delta_b^n}((\Delta^n)^{\diamond})(i) \longrightarrow (\mathbb{S}t_{\Delta_b^n}((\Delta^n)^{\diamond})(i))_{\sharp}$$

and

$$g_n^i\colon \mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^{n-1})^\diamond)(i) \longrightarrow (\mathbb{S}\mathrm{t}_{\Delta^n_\flat}((\Delta^{n-1})^\diamond)(i))_\sharp$$

are MS-anodyne. To show that f_n^i is MS-anodyne it suffices to apply Lemma 3.14. The argument for g_n^i is similar and left as an exercise. We thus obtain, for any i < n a commutative diagram

$$\begin{array}{ccc} \mathbb{S}\mathrm{t}_{\Delta^n_{\flat}}((\Delta^{n-1})^{\diamond})(i) & \stackrel{\phi}{\longrightarrow} \mathbb{S}\mathrm{t}_{\Delta^n_{\flat}}((\Delta^n)^{\diamond})(i) \\ & g \Big| \sim & \sim \Big| f \\ \mathbb{S}\mathrm{t}_{\Delta^n_{\flat}}((\Delta^{n-1})^{\diamond})(i)_{\sharp} & \stackrel{\phi_{\sharp}}{\sim} & \mathbb{S}\mathrm{t}_{\Delta^n_{\flat}}((\Delta^n)^{\diamond})(i)_{\sharp} \end{array}$$

Showing that ϕ is an equivalence of marked-scaled simplicial sets by 2-out-of-3.

Lemma 3.82. Let $X \to \Delta_b^n$ be a 2-Cartesian fibration, and denote by X_i the fibre over i. Let St_*

denote the straightening over Δ^0 . Then the map

$$\psi_i^X : \operatorname{St}_*(X_i) \longrightarrow \operatorname{St}_{\Delta_{\mathfrak{h}}^n}(X)(i)$$

 $is\ an\ equivalence\ of\ marked-scaled\ simplicial\ sets.$

Proof. Following [21, 3.2.3.3], we proceed by induction on n. We have already shown the case n=0 in Corollary 3.50

By construction, ψ_n is an isomorphism. For i < n, we get a canonical commutative diagram

$$\mathbb{S} \mathbf{t}_{\Delta^n_{\flat}}(X|_{\Delta^{n-1}})(i)$$

$$\mathbb{S} \mathbf{t}_*(X_i) \xrightarrow{\psi_i^X} \mathbb{S} \mathbf{t}_{\Delta^n_{\flat}}(X)(i)$$

We can identify the upper-left map with $\psi_i^{X|_{\Delta^{n-1}}}$, and so by the inductive hypothesis, it is an equivalence. It thus suffices for us to show that γ_i is an equivalence.

By Corollary 3.80, we get a homotopy pushout diagram

$$X_n \times (\Delta^{n-1})^{\diamond} \longrightarrow X_n \times (\Delta^n)^{\diamond}$$

$$\downarrow \qquad \qquad \downarrow$$

$$X|_{\Delta^{n-1}} \longrightarrow X$$

in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\Delta_{\flat}^{n}}$. Applying the left Quillen functor $\operatorname{St}_{\Delta_{\flat}^{n}}$ yields a homotopy pushout diagram

$$\mathbb{S}t_{\Delta_{\flat}^{n}}(X_{n}\times(\Delta^{n-1})^{\diamond}) \longrightarrow \mathbb{S}t_{\Delta_{\flat}^{n}}(X_{n}\times(\Delta^{n})^{\diamond})$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathbb{S}t_{\Delta_{\flat}^{n}}(X|_{\Delta^{n-1}}) \longrightarrow \mathbb{S}t_{\Delta_{\flat}^{n}}(X)$$

We have a commutative diagram

$$\mathbb{S} t_{\Delta_{\flat}^{n}}(X_{n} \times (\Delta^{n-1})^{\diamond}) \longrightarrow \mathbb{S} t_{\Delta_{\flat}^{n}}(X_{n} \times (\Delta^{n})^{\diamond})$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathbb{S} t_{*}(X_{n}) \boxtimes \mathbb{S} t_{\Delta_{\flat}^{n}}((\Delta^{n-1})^{\diamond}) \longrightarrow \mathbb{S} t_{*}(X_{n}) \boxtimes \mathbb{S} t_{\Delta_{\flat}^{n}}((\Delta^{n})^{\diamond})$$

where the vertical maps are equivalences of marked-scaled simplicial sets by Theorem 3.17. It thus suffices to note that, by Lemma 3.81, the induced morphism

$$\psi_i \colon \mathbb{S}_{\mathsf{L}^n_b}((\Delta^{n-1})^{\diamond})(i) \longrightarrow \mathbb{S}_{\mathsf{L}^n_b}((\Delta^n)^{\diamond})(i)$$

is an equivalence for any i < n.

Before continuing, we fix some notation to ease the coming discussion. We will in the following theorem denote the straightening-unstraightening equivalence over the point by

$$S: (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\Delta^0} \xrightarrow{\longleftarrow} (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[\Delta^0_{\flat}]^{\mathrm{op}}} : U$$

Proposition 3.83. The Quillen adjunction

$$\mathbb{S}t_{\Delta_{b}^{n}}:(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\Delta_{b}^{n}} \overset{}{\longleftrightarrow} (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[\Delta_{b}^{n}]^{\mathrm{op}}}:\mathbb{U}n_{\Delta_{b}^{n}}$$

is a Quillen equivalence.

Proof. As in [21, Lem. 3.2.3.2], we see that $\mathbb{U}_{\Lambda_{\flat}^n}$ reflects weak equivalences between the images of fibrant objects. It is thus sufficient to show that the derived adjunction unit

$$\mathrm{Id} \Rightarrow \mathsf{R}(\mathbb{U}\mathrm{n}_{\Delta^n_{\mathsf{b}}}) \circ \mathbb{S}\mathrm{t}_{\Delta^n_{\mathsf{b}}}$$

is an equivalence. Since $\mathsf{R}(\mathbb{U}\mathsf{n}_{\Delta^n_\flat})$ preserves weak equivalences and $\mathbb{S}\mathsf{t}_{\Delta^n_\flat}$ preserves trivial cofibrations, it is sufficient to check this for fibrant objects.

Let $X \to \Delta_{\flat}^n$ be a 2-Cartesian fibration, and let

$$\operatorname{St}_{\Delta^n}(X) \xrightarrow{\sim} \mathfrak{F}$$

be a fibrant replacement in $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[\Delta_{\flat}^{n}]^{\mathrm{op}}}$. We are thus left to show that the induced map

$$X \longrightarrow \mathbb{U}\mathrm{n}_{\Delta^n_{\mathsf{h}}}(\mathfrak{F})$$

is an equivalence in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\Delta_{\flat}^{n}}$. Since both objects are fibrant, it suffices to show that this map is a fibrewise equivalence.

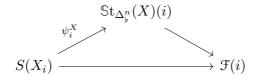
We can identify $\mathbb{U}_{\Delta^n_{\flat}}(\mathcal{F})$ with $U(\mathcal{F}(i))$. Using the equivalence of Corollary 3.50, we see that the map

$$X_i \longrightarrow U(\mathfrak{F}(i))$$

is an equivalence if and only if the adjoint map

$$S(X_i) \longrightarrow \mathfrak{F}(i)$$

is an equivalence. However, we can factor this map as



The upper-right map is an equivalence since \mathcal{F} was a fibrant replacement, and ψ_i^X is an equivalence by Lemma 3.82. The proposition is thus proven.

Corollary 3.84. Consider the scaled simplicial set $(\Delta^2)_{\sharp} := N^{sc}([2])$. Then Quillen adjunction

$$\mathbb{S}t_{\Delta^n_{\sharp}}: (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\Delta^n_{\sharp}} \overset{}{\longleftrightarrow} (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\operatorname{sc}}[\Delta^n_{\sharp}]^{\operatorname{op}}}: \mathbb{U}n_{\Delta^n_{\sharp}}$$

is a Quillen equivalence.

Proof. The key point to note is that base change along the cofibration

$$(\Delta^2)^{\sharp}_{\flat \subset \sharp} \to (\Delta^2)^{\sharp}_{\sharp \subset \sharp}$$

induces a fully faithful inclusion

$$(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\Delta_{\flat}^{2}}^{\circ} \to (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\Delta_{\sharp}^{2}}^{\circ}$$

and similarly, composition with the induced map $\mathfrak{C}^{\mathrm{sc}}[\Delta^2_{\flat}] \to \mathfrak{C}^{\mathrm{sc}}[\Delta^2_{\sharp}]$ induces a fully faithful inclusion

$$\left((\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\operatorname{sc}}[\Delta_{\sharp}^{2}]^{\operatorname{op}}} \right)^{\circ} \to \left((\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\operatorname{sc}}[\Delta_{\flat}^{2}]^{\operatorname{op}}} \right)^{\circ}$$

and so we obtain a commutative diagram

$$\begin{array}{ccc} \left((\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\operatorname{sc}}[\Delta_{\sharp}^{2}]^{\operatorname{op}}} \right)^{\circ} & \xrightarrow{\quad \ \ \, } & \left(\operatorname{Set}_{\Delta}^{\mathbf{mb}} \right)_{/\Delta_{\sharp}^{2}}^{\circ} \\ & & \downarrow & & \downarrow \\ \left((\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\operatorname{sc}}[\Delta_{\flat}^{2}]^{\operatorname{op}}} \right)^{\circ} & \xrightarrow{\quad \ \ \, } & \left(\operatorname{Set}_{\Delta}^{\mathbf{mb}} \right)_{/\Delta_{\flat}^{\circ}}^{\circ} \end{aligned}$$

of simplicial categories.

The remainder of the proof is, mutatis mutandis, that of [23, Prop. 3.8.7].

3.6. Straightening in general

We now prove the main theorem of this paper.

Theorem 3.85. Let $S \in \operatorname{Set}_{\Delta}^{\operatorname{sc}}$ be a scaled simplicial set, and let $\phi : \mathfrak{C}^{\operatorname{sc}}[S] \to \mathcal{C}$ be an equivalence of $\operatorname{Set}_{\Delta}^+$ -enriched categories. The Quillen adjunction

$$\mathbb{S}t_{\phi}: (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S} \Longrightarrow (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathcal{C}^{\operatorname{op}}}: \mathbb{U}n_{\phi}$$

is a Quillen equivalence.

Coupled with the fact, discussed immediately hereafter, that $\mathbb{U}n_{\phi}$ is a $\operatorname{Set}_{\Delta}^+$ -enriched functor, this will immediately imply a stronger result — the functor of $\operatorname{Set}_{\Delta}^+$ -enriched categories of fibrant-cofibrant objects induces an equivalence of ∞ -bicategories.

The argument from here on out is standard, and follows the same path as [23, Section 3.8]. Our first aim will be to show that, for any scaled simplicial set S, the functor

$$\mathbb{U}\mathrm{n}_{\phi}\colon (\mathrm{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\mathrm{sc}}[S]^{\mathrm{op}}} \longrightarrow (\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/S}$$

is, in fact, an $\operatorname{Set}_{\Delta}^+$ -enriched functor.

The $\operatorname{Set}_{\Delta}^+$ -enrichment on Un_{ϕ} is given as follows. Let $\mathcal{F}, \mathcal{G}: \mathcal{C}^{\operatorname{op}} \to \operatorname{Set}_{\Delta}^{\operatorname{ms}}$ be $\operatorname{Set}_{\Delta}^+$ -enriched functors, and $K \in \operatorname{Set}_{\Delta}^+$. A map

$$K \longrightarrow \operatorname{Map}^+(\mathcal{F}, \mathcal{G})$$

is equivalently a map $\mathfrak{F} \otimes K \to \mathfrak{G}$, where $(\mathfrak{F} \otimes K)(s) := \mathfrak{F}(s) \times K_{\sharp}$. We then have a natural map

$$\mathbb{U}\mathrm{n}_{\phi}(\mathfrak{F}) \times K_{\sharp \subset \sharp} \to \mathbb{U}\mathrm{n}_{\phi}(\mathfrak{F}) \times \mathbb{U}\mathrm{n}_{*}(K_{\sharp})$$

Where the second component is induced by the natural transformation $\alpha: St_* \Rightarrow L$. We can then write down a natural composite map

$$\mathbb{U}\mathrm{n}_{\phi}(\mathfrak{F}) \times K_{\mathfrak{h}\mathfrak{C}\mathfrak{h}} \to \mathbb{U}\mathrm{n}_{S}(\mathfrak{F}) \times \mathbb{U}\mathrm{n}_{*}(K_{\mathfrak{h}}) \to \mathbb{U}\mathrm{n}_{\phi}(\mathfrak{F} \otimes K) \to \mathbb{U}\mathrm{n}_{\phi}(\mathfrak{F})$$

Which is equivalently a map $K \to \operatorname{Map}^{\operatorname{th}}(\operatorname{Un}_S(\mathcal{F}), \operatorname{Un}_S(\mathcal{G}))$. The naturality guarantees that this defines a map of simplicial sets

$$\operatorname{Map}^+(\mathfrak{F},\mathfrak{G}) \to \operatorname{Map}^{\operatorname{th}}(\operatorname{Un}_{\phi}(\mathfrak{F}),\operatorname{Un}_{\phi}(\mathfrak{G})).$$

Similarly, since the composition maps in both cases are defined via the diagonal $\Delta^n \to \Delta^n \times \Delta^n$,

naturality ensures that this defines an enriched functor. A wholly analogous argument shows that \mathbb{U}_{n_S} can also be viewed as a simplicially-enriched functor.

Proof (of Theorem 3.85). The proof is now nearly identical to that of [23, Prop. 3.8.4]. The argument hangs on the claim that the functor

$$F : (\operatorname{Set}_{\Delta}^{\operatorname{\mathbf{sc}}})^{\operatorname{op}} \longrightarrow \operatorname{Cat}_{\Delta}$$

$$S \longmapsto ((\operatorname{Set}_{\Delta}^{\operatorname{\mathbf{ms}}})_{f}^{\operatorname{\mathfrak{C}}^{\operatorname{\mathbf{sc}}}[S]^{\operatorname{op}}})[W_{S}^{-1}]$$

sends pushouts along cofibrations to homotopy pullbacks, and sends transfinite composites of cofibrations to homotopy limits, which follows from the argument given in loc. cit. \Box

Corollary 3.86. Let $S \in \operatorname{Set}_{\Delta}^{\mathbf{sc}}$ be an ∞ -bicategory. The $\operatorname{Set}_{\Delta}^+$ -enriched functor \mathbb{U}_{n_S} induces an equivalence of ∞ -bicategories

$$N^{\operatorname{sc}}\left(\left((\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathfrak{C}^{\operatorname{sc}}[S]^{\operatorname{op}}}\right)^{\circ}\right) \longrightarrow N^{\operatorname{sc}}\left(\left((\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}\right)^{\circ}\right).$$

Proof. This follows immediately from Theorem 3.17, Theorem 3.85, and [21, A.3.1.10].

One final step is left: to interpret this result internally to marked-scaled simplicial sets.

Definition 3.87. The $\operatorname{Set}_{\Delta}^+$ -enrichment on $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ equips the full subcategory $(\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\circ}$ of fibrant-cofibrant objects with the structure of a fibrant $\operatorname{Set}_{\Delta}^+$ -enriched category. We denote by $\operatorname{Bicat}_{\infty} := \operatorname{N}^{\operatorname{sc}}((\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\circ})$ the homotopy-coherent scaled nerve of this $\operatorname{Set}_{\Delta}^+$ -category (considered as a scaled simplicial set). We refer to $\operatorname{Bicat}_{\infty}$ as the ∞ -bicategory of ∞ -bicategories.

Similarly, for $S \in \operatorname{Set}_{\Delta}^{\mathbf{sc}}$, we denote by $2\mathbb{C}\operatorname{art}(S) := \operatorname{N}^{\operatorname{sc}}\left(\left((\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/S}\right)^{\circ}\right)$ the ∞ -bicategory of 2-Cartesian fibrations over S.

Remark 3.88. Formally, considering $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$ as the category of all \mathcal{U} -small marked-scaled simplicial sets for some Grothendieck universe \mathcal{U} , the marked-scaled simplicial set $\mathbb{B}\mathrm{icat}_{\infty}$ is no longer small. We thus resort to fixing a new Grothendieck universe \mathcal{V} in which \mathcal{U} , and thus $\mathbb{B}\mathrm{icat}_{\infty}$, becomes \mathcal{V} -small.

Proposition 3.89. Let \mathcal{C} be a small $\operatorname{Set}_{\Delta}^+$ -enriched category, S a small scaled simplicial set, ϕ : $\mathfrak{C}^{\operatorname{sc}}[S] \to \mathcal{C}$ an equivalence of $\operatorname{Set}_{\Delta}^+$ -enriched categories, and \mathbf{A} a combinatorial, $\operatorname{Set}_{\Delta}^+$ -enriched model category. Endow $\mathbf{A}^{\mathcal{C}}$ with the projective model structure. Then the functor

$$N^{sc}((\mathbf{A}^{\mathfrak{C}})^{\circ}) \to \operatorname{Fun}(S, N^{sc}(\mathbf{A}^{\circ}))$$

is a bicategorical equivalence of scaled simplicial sets.

Proof. The proof is that of [21, Prop. 4.2.4.4]. The only thing that changes is the exchange of $\operatorname{Set}_{\Delta}$ for $\operatorname{Set}_{\Delta}^{\mathbf{ms}}$, and as both of these are excellent model categories, no further emendation is necessary.

Corollary 3.90. Let $S \in \operatorname{Set}_{\Delta}^{\operatorname{sc}}$. There is an equivalence of ∞ -bicategories

$$2\mathbb{C}\operatorname{art}(S) \simeq \operatorname{Fun}(S^{\operatorname{op}}, \mathbb{B}\operatorname{icat}_{\infty}).$$

4. Application: Marked colimits and higher cofinality

In previous work [1], [2] we have studied 2-categorical notions of (co)limits in ∞ -bicategories. In this section we will prove the cofinality conjecture as stated in [2]. Let us remark that in previous work we only considered specific instances of colimits:

- In [2] we considered diagrams $F: \mathbb{D}^{\dagger} \to \mathbb{A}$ where both \mathbb{D} and \mathbb{A} are strict 2-categories and \mathbb{D}^{\dagger} comes equipped with a collection of marked edges.
- In [1] we considered diagrams $F: \mathcal{D}^{\dagger} \to \mathbb{A}$ where \mathcal{D}^{\dagger} is an ∞ -category equipped with a collection of marked edges and \mathbb{A} is an ∞ -bicategory.

The general notion of marked colimit has been extensively studied by Gagna, Harpaz and Lanari in [11]. It was noted by the authors that their notion coincides with ours in the cases we studied as shown in [11, Remark 5.2.12.]. Moreover, the notion of cofinality studied in [1] also agrees with the definition of cofinal functor given in [11]. However, in the previous document no computational criterion is given to determine whether a functor of marked ∞ -bicategories $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ is cofinal. We will provide such criterion as the main result in this section extending the well-known conditions of Quillen's Theorem A.

4.1. The free 2-Cartesian fibration

Throughout this section, we fix an ∞ -bicategory \mathbb{D} , and aim to construct, for each functor of ∞ -bicategories $p: \mathbb{X} \to \mathbb{D}$, a 2-Cartesian fibration $\mathbb{F}(p): \mathbb{F}(\mathbb{X}) \to \mathbb{D}$. We will characterize this fibration as the free 2-Cartesian fibration on p.

To construct this 2-Cartesian fibration, we will make use of the Gray tensor products constructed in [9, §2.1] and [11, §4.1]. To this end, we briefly recall the definitions we will need from [9].

Definition 4.1. Let $X, Y \in \operatorname{Set}_{\Delta}^{\operatorname{sc}}$. We define the *Gray product* $X \otimes Y$ to be the scaled simplicial set with underlying simplicial set $X \times Y$, where we declare a 2-simplex (σ_X, σ_Y) to be scaled if and only if the following two conditions are both satisfied

- $(\sigma_X, \sigma_Y) \in T_X \times T_Y$.
- Either the image of σ_X degenerates along $\Delta^{\{1,2\}}$ or σ_Y degenerates along $\Delta^{\{0,1\}}$.

Given scaled simplicial sets X and Y, we define the Gray functor category $\operatorname{Fun}^{\operatorname{gr}}(X,Y)$ by the adjunction

$$\operatorname{Hom}_{\operatorname{Set}^{\mathbf{sc}}_{\Lambda}}(S, \operatorname{Fun}^{\operatorname{gr}}(X, Y)) \cong \operatorname{Hom}_{\operatorname{Set}^{\mathbf{sc}}_{\Lambda}}(X \otimes S, Y).$$

There is a dual version, defined by replacing $X \otimes S$ by $S \otimes X$, which we denote by Fun^{opgr}(X, Y).

Proposition 4.2. Let \mathbb{D} an ∞ -bicategory. Then the map ev_0 : $\operatorname{Fun}^{\operatorname{gr}}(\Delta^1, \mathbb{D}) \longrightarrow \mathbb{D}$ is a 2-Cartesian fibration. The collection of Cartesian edges, thin triangles, and lean (coCartesian) triangles can be described as follows:

- An edge represented by a map $e: \Delta^1 \otimes \Delta^1 \to \mathbb{D}$ is Cartesian if and only if it factors through $\Delta^1 \times \Delta^1$ and the restriction to $\Delta^{\{1\}} \times \Delta^1$ is an equivalence in \mathbb{D} .
- A triangle represented by a map $\sigma: \Delta^1 \otimes \Delta^2 \to \mathbb{D}$ is lean if and only if its restriction to $\Delta^{\{1\}} \times \Delta^2$ is thin in \mathbb{D} .
- A triangle represented by a map $\sigma: \Delta^1 \otimes \Delta^2 \to \mathbb{D}$ is thin if and only if it is lean and its restriction to $\Delta^{\{0\}} \times \Delta^2$ is thin.

Proof. Since, over an ∞ -bicategory, the definition of 2-Cartesian we provided in [4] coincides with the notion of *outer 2-Cartesian fibration* from [12], this follows immediately from [12, Thm 2.2.6].

We can leverage this definition to give an extension of the Gray product, which more fully captures the decoration in this case.

Definition 4.3. Let $X \in \operatorname{Set}_{\Delta}^{\mathbf{mb}}$ and denote by \widetilde{X} its underlying scaled simplicial set. We define $\Delta^1 \widehat{\otimes} X \in \operatorname{Set}_{\Delta}^{\mathbf{ms}}$ extending $\Delta^1 \otimes \widetilde{X}$ by declaring

- A 1-simplex (σ_1, σ_X) is marked if it is degenerate, or if σ_1 is degenerate on $\{1\}$, and σ_X is marked in X.
- A 2-simplex (σ_1, σ_X) is thin if any of the following conditions hold.
 - The simplex (σ_1, σ_X) is thin in $\Delta^1 \otimes \widetilde{X}$.
 - The simplex σ_X is lean in X and $\sigma_1(1 \to 2)$ is degenerate on 1.
 - The simplex σ_X is lean in X, $\sigma_X(0 \to 1)$ is marked in X and the simplex σ_1 is of the form $0 \to 0 \to 1$.

For $X, Y \in \operatorname{Set}_{\Delta}^{\mathbf{ms}}$, we can then define $\operatorname{Fun}^{\widehat{\operatorname{gr}}}(\Delta^1, Y) \in \operatorname{Set}_{\Delta}^{\mathbf{mb}}$ via the adjunction

$$\operatorname{Hom}_{\operatorname{Set}_{\Delta}^{\mathbf{mb}}}(S, \operatorname{Fun}^{\widehat{\operatorname{gr}}}(\Delta^1, Y)) \cong \operatorname{Hom}_{\operatorname{Set}_{\Delta}^{\mathbf{ms}}}(\Delta^1 \otimes S, Y).$$

Remark 4.4. For $X = (X, M_X, T_X \subset C_X)$, we will denote by $\{0\} \widehat{\otimes} X$ the full **MS** simplicial subset of $\Delta^1 \widehat{\otimes} X$ corresponding to $\{0\} \times X$. This is isomorphic to the **MS** simplicial set (X, \flat, T_X) . Similarly, we denote by $\{1\} \widehat{\otimes} X$ the **MS** simplicial set (X, M_X, C_X) .

We can then define the free 2-Cartesian fibration.

Definition 4.5. Let $p: \mathbb{X} \to \mathbb{D}$ be a functor of ∞ -bicategories. Denote by $\mathbb{X}^{\natural} = (\mathbb{X}, M_X, T_X \subset T_X)$ the associated \mathbf{MB} simplicial set in which the equivalences are marked. We define an \mathbf{MB} simplicial set $\mathbb{F}(\mathbb{X})^{\natural}$ as the pullback

$$\begin{array}{ccc}
\mathbb{F}(\mathbb{X})^{\natural} & \longrightarrow & \operatorname{Fun}^{\widehat{\operatorname{gr}}}(\Delta^{1}, \mathbb{D}) \\
\downarrow & & \downarrow^{\operatorname{ev}_{1}} \\
\mathbb{X}^{\natural} & \longrightarrow & \mathbb{D}
\end{array}$$

We denote the natural map induced by evaluation at 0 by $\mathbb{F}(p) \colon \mathbb{F}(\mathbb{X})^{\natural} \longrightarrow \mathbb{D}$.

Proposition 4.6. Let $p: \mathbb{X} \to \mathbb{D}$ be a functor of ∞ -bicategories. Then

$$\mathbb{F}(p) \colon \mathbb{F}(\mathbb{X})^{\natural} \longrightarrow \mathbb{D}$$

is a 2-Cartesian fibration

This proposition will follow from a somewhat technical lemma.

Lemma 4.7. Let $f: A \longrightarrow B$ be an **MB**-anodyne morphism. Then

$$\Delta^1 \widehat{\otimes} A \coprod_{\{0\} \widehat{\otimes} A} \{0\} \widehat{\otimes} B \longrightarrow \Delta^1 \widehat{\otimes} B$$

is MS-anodyne.

Proof. As usual, we can check on generating **MB**-anodyne morphisms. Before commencing the proof, we will make a preliminary definition. We say that a morphism of **MS** simplicial sets is of type (\heartsuit) if it is in the weakly saturated hull of morphisms of the type described in Lemma 2.41. We can now proceed to perfom our case-by-case analysis.

- (A2) We can scale $\{1\} \widehat{\otimes} \Delta^4$ using a pushout of type (MS2). The remaining 2-simplices can be scaled using morphisms of type (\heartsuit).
- (A5) Is a pushout of type (MS1), a pushout of type (MS5), and a pushout of type (MS4).
- (S1) Is an iterated pushout of type (MS6) and morphisms of type (\heartsuit).

- (S2) Is an isomorphism.
- (S3) Is a pushout along a morphism of type (\heartsuit) .
- (S4) Is a pushout along morphisms of type (MS7) and (\heartsuit) .
- (S5) Is a pushout along morphisms of type (MS8) and (\heartsuit) .

The remaining three cases are the horn inclusions.

- (A1) Since no morphisms are marked on either side and the lean and thin scalings are identical, this is a consequence of [11, Prop 4.1.9].
- (A4) Let us set the notation $(\Delta^n)^{\dagger} = (\Delta^n, \Delta^{\{n-1,n\}}, \flat \subset \Delta^{\{0,n-1,n\}})$ and let us similarly define $(\Lambda_n^n)^{\dagger}$. First we will define an order for the simplices of maximal dimension in $\Delta^1 \widehat{\otimes} (\Delta^n)^{\dagger}$. Let $\theta_{\varepsilon} : \Delta^{n+1} \to \Delta^1 \widehat{\otimes} (\Delta^n)^{\dagger}$ for $\varepsilon \in \{0,1\}$ and let $\nu_{\theta_{\varepsilon}}$ be the first element in Δ^{n+1} such that the value of θ_{ε} at $\nu_{\theta_{\varepsilon}}$ has the first coordinate equal to 1. We say that $\theta_1 < \theta_2$ if and only if $\nu_{\theta_1} < \nu_{\theta_2}$. We further observe every simplex of maximal dimension is uniquely determined by the value ν_{θ} . Consequently we will denote by θ_i for $i \in \{1, 2, ..., n+1\}$ the unique simplex of maximal dimension that has $\nu_{\theta_i} = i$. We can produce now a filtration

$$\Delta^1 \widehat{\otimes} (\Lambda_n^n)^{\dagger} \to Y_{n+1} \to Y_n \to \cdots \to Y_2 \to Y_1 = \Delta^1 \widehat{\otimes} (\Delta^n)^{\dagger}$$

where Y_j is the full **MS** subsimplicial set of $\Delta^1 \widetilde{\otimes} (\Delta^n)^\dagger$ containing the simplices of Y_{j+1} in addition to the simplex θ_j . It is an straightforward to see that the first map is a pushout along the inner-horn inclusion $\Lambda_n^{n+1} \to \Delta^{n+1}$. Since the edge $n-1 \to n$ is marked in $(\Delta^n)^\dagger$ it follows that the triangle $\{n-1,n,n+1\}$ is thin in θ_{n+1} . The rest of the morphisms are in the weakly satured hull of morphisms of type (MS4). Each map $Y_{j+1} \to Y_j$ is obtained precisely after taking two pushouts of type (MS4). First we had the face missing j-1 of θ_j and then we add the whole simplex missing.

(A3) The argument here is precisely dual to the previous case, replacing 'marked edge" with "degenerate edge" and "(MS4)" with "(MS3)". Note that we also need to reverse the order in which the add the simplices of maximal dimension that are missing.

Proof (of 4.6). Given a lifting problem

$$\begin{array}{ccc}
A & \longrightarrow & \mathbb{F}(\mathbf{X})^{\natural} \\
f \downarrow & & \downarrow \\
B & \longrightarrow & \mathbb{D}
\end{array}$$

where f is **MB**-anodyne, we need to find a diagram

$$\begin{cases}
1 \\ \widehat{\otimes} B & \longrightarrow \mathbf{X} \\
\downarrow p \\
\Delta^1 \widehat{\otimes} B & \longrightarrow \mathbf{D}
\end{cases}$$

extending the diagram

$$\begin{cases} 1 \} \widehat{\otimes} A & \longrightarrow \mathbf{X} \\ \downarrow & \downarrow p \\ \Delta^1 \widehat{\otimes} A & \coprod_{\{0\} \widehat{\otimes} A} \{0\} \widehat{\otimes} B & \longrightarrow \mathbf{D} \end{cases}$$

defined by the lifting problem. We first note that, since $\{1\} \widehat{\otimes} A \to \{1\} \widehat{\otimes} B$ is, in particular, **MS**-anodyne, we can solve the lifting problem on the bottom. It thus remains for us to solve the extension problem

However, the morphism on the left fits into a composite.

$$\Delta^1 \widehat{\otimes} A \coprod_{\{0\} \widehat{\otimes} A} \{0\} \widehat{\otimes} B \longrightarrow \{1\} \widehat{\otimes} B \coprod_{\{1\} \widehat{\otimes} B} \Delta^1 \widehat{\otimes} A \coprod_{\{0\} \widehat{\otimes} A} \{0\} \widehat{\otimes} B \longrightarrow \Delta^1 \widehat{\otimes} B$$

where the first morphism is a pushout by an MS-anodyne morphisms, and the composite is one of the morphisms from Lemma 4.7. Consequently, the morphism on the left is an MS trivial cofibration by 2-out-of-3, and thus the lifting problem has a solution.

Definition 4.8. Let $p: \mathbb{X} \to \mathbb{D}$ be a functor of ∞ -bicategories. We denote by $\mathbb{X}_{d\uparrow}$ the fibre of $\mathbb{F}(p): \mathbb{F}(\mathbb{X})^{\natural} \longrightarrow \mathbb{D}$ over $d \in \mathbb{D}$. Note that Proposition 4.6 implies that $\mathbb{X}_{d\uparrow}$ is an ∞ -bicategory.

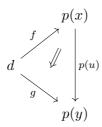
Remark 4.9. Unwinding the definition of $X_{d\uparrow}$, we see that a morphism from $f: d \to p(x)$ to $g: d \to p(y)$ is given by a diagram

$$d \xrightarrow{f} p(x)$$

$$\downarrow p(u)$$

$$d \xrightarrow{g} p(y)$$

which, in a strict 2-category, we could view as a diagram



which justifies our notation. We will later see that, for a functor $F:\mathbb{C}\to\mathbb{D}$ of strict 2-categories, there is an equivalence of ∞ -bicategories

$$N^{\mathrm{sc}}(\mathbb{C}_{d^{\uparrow}}) \simeq N^{\mathrm{sc}}(\mathbb{C})_{d^{\uparrow}}.$$

connecting our free fibration to more familiar notions.

Remark 4.10. Let $p: \mathbb{X} \to \mathbb{D}$ be a functor of ∞ -bicategories. There is a cofibration

$$\gamma_{\mathbf{X}} \colon \mathbf{X} \longrightarrow \mathbb{F}(\mathbf{X})^{\natural}$$

over \mathbb{D} , which sends a simplex $\Delta^n \to \mathbb{X}$ to the map $\Delta^1 \widehat{\otimes} \Delta^n \to \Delta^0 \otimes \Delta^n \to \mathbb{X} \to \mathbb{D}$.

To simplify our examination of this map, we provide a way of constructing 'almost degenerate' (n+1)-simplices in $\mathbb{F}(\mathbf{X})^{\natural}$ from n-simplices in $\mathbb{F}(\mathbf{X})^{\natural}$.

Construction 4.11. For every $0 \le j \le n$, we define a map

$$E_j \colon \Delta^1 \times \Delta^{n+1} \longrightarrow \Delta^1 \times \Delta^n$$

$$(m,r) \longmapsto \begin{cases} (m,r) & r \leqslant j \\ (1,r-1) & r > j \end{cases}$$

It is easy to check that this map respects scalings, and thus yields $E_j: \Delta^1 \otimes \Delta^{n+1} \to \Delta^1 \otimes \Delta^n$. Moreover, the induced map $\{1\} \times \Delta^{n+1} \to \{1\} \times \Delta^n$ is precisely the degeneracy map s_j .

Given an *n*-simplex $\sigma: \Delta^n \to \mathbb{F}(\mathbb{X})$ (possibly having some non-trivial decorations) defined by $\phi^{\sigma}: \Delta^1 \widehat{\otimes} \Delta^n \to \mathbb{D}$ and $\rho^{\sigma}: \{1\} \times \Delta^n \to \mathbb{X}$, we define an (n+1)-simplex $E_i^*(\sigma)$ by

$$\Delta^{1} \otimes \Delta^{n+1} \xrightarrow{E_{j}} \Delta^{1} \widehat{\otimes} \Delta^{n} \xrightarrow{\phi^{\sigma}} \mathbb{D}$$

$$\uparrow \qquad \qquad \uparrow \qquad \qquad \uparrow^{p}$$

$$\Delta^{n+1} \times \{1\} \xrightarrow{s_{j}} \{1\} \times \Delta^{n} \xrightarrow{\rho^{\sigma}} \mathbb{X}$$

We will call $E_j^*(\sigma)$ the j^{th} extention of σ .

Given a simplex $\sigma: \Delta^n \to \mathbb{F}(\mathbb{X})$ as above, we will denote by ℓ_0^{σ} the corresponding map $\{1\} \times \Delta^n \to \mathbb{X}$.

The following lemmata follow immediately from the definition.

Lemma 4.12. Let $\sigma: \Delta^n \to \mathbb{F}(\mathbb{X})$. Then the faces of $E_i^*(\sigma)$ can be written as follows.

- If j = n, and s = n + 1, then $d_s(E_i^*(\sigma)) = \sigma$.
- If $j + 1 < s \le n + 1$, then $d_s(E_j^*(\sigma)) = E_j^*(d_{s-1}(\sigma))$.
- If $0 \le s < j$, then $d_s(E_i^*(\sigma)) = E_{j-1}(d_s(\sigma))$.
- If s = j + 1, then $d_{j+1}(E_i^*(\sigma)) = d_{j+1}(E_{j+1}^*(\sigma))$.
- If s=j and $s\neq 0$, then $d_i(E_i^*(\sigma))=d_i(E_{i-1}^*(\sigma))$
- If s = j = 0, then $d_0(E_0^*(\sigma)) = \gamma_{\mathbf{X}}(d_0(\ell_0^{\sigma}))$.

Lemma 4.13. If $\sigma: \Delta^n \to \mathbb{F}(\mathbb{X})$ is degenerate, then for every $0 \leq j \leq n$, the j^{th} extension $E_j^*(\sigma)$ is degenerate.

Lemma 4.14. Let $\sigma: \Delta^{n-1} \to \mathbb{F}(\mathbb{X})$. Then for every $0 \leqslant j \leqslant n-1$ and every $0 \leqslant i \leqslant n$, the simplex $E_i^*(E_j^*(\sigma))$ is degenerate.

Theorem 4.15. Let $p: \mathbb{X} \longrightarrow \mathbb{D}$ be a functor of ∞ -bicategories. Then the morphism

$$\gamma_{\mathbf{X}} \colon \mathbf{X}^{
atural} \longrightarrow \mathbb{F}(\mathbf{X})^{
atural}$$

is MB-anodyne over \mathbb{D} .

Proof. Let us start by defining Z_0 to be the subsimplicial set of $\mathbb{F}(\mathbf{X})$ consisting of all of the simplices belonging to \mathbf{X} , all of the 0-simplices of $\mathbb{F}(\mathbf{X})$ and all of the possible j-extensions of the 0-simplices. We extend this definition inductively by defining Z_n to consist of all of the simplices of Z_{n-1} , all of the n-simplices of $\mathbb{F}(\mathbf{X})$ and the (n+1)-simplices appearing as extensions of n-simplices. We set the convention $Z_{-1} = X$ and we fix the notation

$$\gamma_{n-1}\colon Z_{n-1}\longrightarrow Z_n.$$

Observe that since **MB**-anodyne maps are stable under transfinite composition it will suffice to show that each γ_{n-1} is **MB**-anodyne.

We start by analyzing γ_{-1} . Observe that given an object $\sigma: \Delta^0 \to \mathbb{F}(\mathbf{X})$ we can consider the Cartesian edge $E_0^*(\sigma): \Delta^1 \to \mathbb{F}(\mathbf{X})$. Since the target of $E_0^*(\sigma)$ is already contained in \mathbf{X} it follows that we can add σ by means of a pushout along a \mathbf{MB} -anodyne map. Repeating this process for each object in $\mathbb{F}(\mathbf{X})$ conclude that γ_{-1} is \mathbf{MB} -anodyne.

Now we will tackle the general case for γ_{n-1} with $n \ge 1$. Let us pick an order on the set of non-degenerate n-simplices of $\mathbb{F}(\mathbf{X})$ that are not already contained in Z_{n-1} . For every $\sigma \in \mathbb{F}(\mathbf{X})_n$ we define $Z_{n-1}(\sigma)$ as the subsimplicial subset of Z_n containing all of the simplices of Z_{n-1} in addition to the n-simplices $\theta \le \sigma$ and its corresponding extensions. Let $\mathrm{suc}(\sigma)$ be the successor of σ in our chosen order. We will adopt the convention $Z_{n-1}(\varnothing) = Z_{n-1}$ and $\mathrm{suc}(\varnothing) = \sigma_0$ is the first element in our ordering. To show that γ_{n-1} is **MB**-anodyne it will suffice to prove that

$$Z_{n-1}(\sigma) \longrightarrow Z_{n-1}(\rho)$$

is **MB**-anodyne where $\rho = \operatorname{suc}(\sigma)$.

The proof will be divided into three cases. First let us assume that for every $1 \le j \le n$ all of the faces of $E_j^*(\rho)$ are contained in $Z_{n-1}(\sigma)$ except the faces missing j+1,j. Applying Lemma 4.12 for j=0 yields

$$d_s E_0^*(\rho) = \begin{cases} E_0^* (d_s(\rho)), & \text{if } 1 < s \le n+1 \\ d_1 (E_1^*(\rho)), & \text{if } s = 1 \\ \gamma_X (d_0(\ell_0)), & \text{if } s = 0. \end{cases}$$

which shows that all of the faces of $E_0^*(\rho)$ are already in $Z_{n-1}(\sigma)$ except the 1-face. By construction the triangle $\Delta^{\{0,1,2\}}$ is thin in $E_0^*(\rho)$, which shows that we can add the simplex $E_0^*(\rho)$ via a pushout along a **MB**-anodyne map. Let us denote by V_0 the resulting simplicial set $Z_{n-1}(\sigma) \to V_0 \to Z_{n-1}(\rho)$. Using Lemma 4.12 again, we see that all of the faces of $E_1^*(\rho)$ are in V_0 except the 2-face. A similar argument as above shows that we can add $E_1^*(\rho)$ in a **MB**-anodyne way and thus obtaining a new subsimplicial set that we denote V_1 . We can repeat this process until we reach V_{n-1} . In our final step we observe that we have a pullback diagram

$$\Lambda_{n+1}^{n+1} \longrightarrow \Delta^{n+1}
\downarrow \qquad \qquad \downarrow E_n^*(\rho)
V_{n-1} \longrightarrow Z_{n-1}(\rho)$$

where the last edge of Λ_{n+1}^{n+1} is 2-Cartesian in $\mathbb{F}(\mathbf{X})$ and the triangle $\Delta^{\{0,n,n+1\}}$ is coCartesian. Therefore we can add $E_n^*(\rho)$ using a pushout along a **MB**-anodyne map and conclude that $Z_{n-1}(\sigma) \to Z_{n-1}(\rho)$ is in this case **MB**-anodyne.

For the second case let us suppose that there exists some $1 \leqslant \alpha \leqslant n$ such that $d_{\alpha}(E_{\alpha}^{*}(\rho))$ is already in $Z_{n-1}(\sigma)$ and that for every $j > \alpha$ we have that the faces missing j+1, j in $E_{j}^{*}(\rho)$ are not contained in $Z_{n-1}(\sigma)$. We claim that for every $0 \leqslant k < \alpha$ the simplex $E_{k}^{*}(\rho) \in Z_{n-1}(\rho)$. One can easily check that

$$E_k^*(\rho) = E_k^*(d_\alpha(E_\alpha^*(\rho))), \quad 0 \leqslant k < \alpha.$$

In particular, this shows that it will suffice to show that all of the extensions of $d_{\alpha}(E_{\alpha}^{*}(\rho))$ are contained in $Z_{n-1}(\sigma)$. To provide a proof of this later claim we observe that $d_{\alpha}(E_{\alpha}^{*}(\rho)) \in Z_{n-1}(\sigma)$ if and only at least one of the following conditions hold:

- *) The face $d_{\alpha}(E_{\alpha}^{*}(\rho))$ is contained in **X**.
- i) The face $d_{\alpha}(E_{\alpha}^{*}(\rho))$ is degenerate.
- ii) The face $d_{\alpha}(E_{\alpha}^{*}(\rho))$ is the extension of an (n-1)-simplex.

iii) There exists $\theta \leqslant \sigma$, such that $d_{\alpha}(E_{\alpha}^{*}(\rho))$ is a face of an extension of θ

If condition *) holds then it is easy to see that all of the possible extensions are already in $Z_{n-1}(\sigma)$. Using Lemma 4.13 and Lemma 4.14 we see that the claim holds if the conditions i) or ii) are satisfied. Suppose now that condition iii) holds. We can assume without loss of generality that $d_{\alpha}(E_{\alpha}^{*}(\rho)) = d_{\beta}(E_{\beta}^{*}(\theta))$. A straightforward computation shows that

$$E_j^*(d_{\beta}(E_{\beta}^*(\theta))) = \begin{cases} s_j(d_{\beta}(E_{\beta}^*(\theta))), & \text{if } j \geqslant \beta \\ E_j^*(\theta), & \text{if } j < \beta \end{cases}$$

so again, the claim holds. We have shown $Z_{n-1}(\sigma) = V_{\alpha-1}$ and thus the previous argument runs exactly the same way.

The last case to analyze is the degenerate situation where ρ is already in $Z_{n-1}(\sigma)$. In this case we need to show that $Z_{n-1}(\sigma) = Z_{n-1}(\rho)$, i.e. we need to show that we already have all of the extensions of ρ . Since $\rho \notin Z_{n-1}$ and it is not degenerate it follows that $\rho = d_{\beta}(E_{\beta}(\theta))$ for some $\theta \leqslant \sigma$. Using the same reasoning as before we can see that $E_k(\rho) \in Z_{n-1}(\sigma)$ for all $0 \leqslant k \leqslant n$ and the claim follows.

In order to finish the proof there is one last thing we have to take care of in the filtration, namely, the decorations. We need to show that whenever we add a marked edge (resp. lean, resp. thin triangle) in our filtration we can add the decoration to our filtration in a MB-anodyne way. For the marked edges this essentially an specific case of the proof given in Corollary 4.18. We leave the rest of the decorations as an exercise for the reader.

Remark 4.16. The morphism $\gamma_X : X^{\natural} \to \mathbb{F}(X)^{\natural}$ we can viewed as the unit of a bicategorical free-forgetful adjunction between the ∞ -bicategory of of ∞ -bicategories over \mathbb{D} and the ∞ -bicategory of 2-Cartesian fibrations over \mathbb{D} . We will not pursue this direction further in this document. A detailed study of bicategorical adjunctions is part of the research program of the authors and will appear in future work.

Definition 4.17. Let $p: \mathbb{X} \longrightarrow \mathbb{D}$ be a functor of ∞ -bicategories. Assume that \mathbb{X} comes equipped with a marking containing all of the equivalences and denote the resulting marked ∞ -bicategory by \mathbb{X}^{\dagger} . We define new marking on $\mathbb{F}(\mathbb{X})$ as follows. We declare and edge represented by $\Delta^1 \otimes \Delta^1 \to \mathbb{D}$ marked if and only if it factors through $\Delta^1 \times \Delta^1$ and its restriction to $\Delta^{\{1\}} \times \Delta^1$ factors through a marked edge in \mathbb{X} . We define marked-scaled simplicial $\mathbb{F}(\mathbb{X})^{\dagger}$ having the same lean and thin triangles as $\mathbb{F}(\mathbb{X})^{\natural}$ but equipped with this new collection of marked edges.

Corollary 4.18. Let $p: \mathbb{X} \longrightarrow \mathbb{D}$ be a functor of ∞ -bicategories. Assume that X comes equipped with a marking (containing the equivalences) and denote the corresponding marked ∞ -bicategory by \mathbb{X}^{\dagger} . Then the morphism

$$X^{\dagger} \longrightarrow \mathbb{F}(X)^{\dagger}$$

is MB-anodyne.

Proof. Let us consider the pushout diagram

$$egin{array}{cccc} old X & \longrightarrow & old X^\dagger & & & \downarrow & \downarrow & & \downarrow & & \downarrow & \downarrow & & \downarrow & \downarrow$$

where it follows from Theorem 4.15 that the left-most vertical map is MB-anodyne. To finish the proof we just need to show that the morphism $\mathbb{F}(\mathbf{X})^{\diamond} \to \mathbb{F}(\mathbf{X})^{\dagger}$ is again anodyne. Let $e: \Delta^1 \otimes \Delta^1 \to \mathbb{D}$ be a marked edge of $\mathbb{F}(\mathbf{X})^{\dagger}$. First we observe that $E_0^*(e)$ is a thin 2-simplex such that $d_0(E_0^*(e))$ and $d_2(E_0^*(e))$ are marked in $\mathbb{F}(\mathbf{X})^{\diamond}$. It particular it follows that we can marked the edge $d_1(E_0^*(e))$

using a pushout along a morphism of type (S1). Using a pushout along a morphism of the type described in [4, Lem. 3.7] we can mark all edges in $E_1^*(e)$. We conclude the proof after noting that $d_2(E_1^*(e)) = e$.

4.2. Marked colimits and cofinality

We now turn to our main result of this section, a criterion for higher cofinality. We will not here recapitulate the theory of higher (co)limits expounded in [11], but see Remark 4.20 for details on the connection with $(\infty, 2)$ -categorical colimits.

Definition 4.19. Let X^{\dagger}, Y^{\dagger} be a pair of marked-scaled simplicial sets and consider a marking preserving functor, $f: X^{\dagger} \to Y^{\dagger}$. We say that f is a marked cofinal functor if the associated functor of marked-biscaled simplicial sets

$$f: (X, E_X, T_X \subset \sharp) \longrightarrow (Y, E_Y, T_Y \subset \sharp)$$

is a weak equivalence in model structure of MB simplicial sets over Y.

Remark 4.20. The theory of marked colimits in ∞ -bicategories was independently developed by Berman in [6], the present authors in [3] and [1], and Gagna, Harpaz, and Lanari in [11]. The latter provides a full characterization of marked (co)limits in ∞ -bicategories, including the four variances which arise from changing the directions of 2-morphisms in the corresponding notion of cone. The theory of marked colimits described in [1] corresponds to the case of *outer colimits* in the language of [11].

By [11, Thm 5.4.4], a functor of marked ∞ -bicategories $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ is outer cofinal — i.e., pullback along f preserves outer colimits — if and only if f is marked cofinal in the sense of Definition 4.19 above (compare [11, Defn 4.3.3] to [4, Defn 3.25] and [4, Prop. 3.28] to see that these conditions do indeed coincide).

Remark 4.21. Let $f:(\mathbb{C}, E_{\mathbb{C}}, T_{\mathbb{C}}) \to (\mathbb{D}, E_{\mathbb{D}}, T_{\mathbb{D}})$ be a functor of marked ∞ -bicategories. Observe that in order to see if f is cofinal we can assume that the markings of both ∞ -bicategories contain all equivalences. Indeed, this follows easily after taking pushouts along morphisms of type (E). Consequently for the rest of the section we will assume that the markings satisfy this property.

Lemma 4.22. Let $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ be a functor of marked ∞ -bicategories and recall from Definition 4.17 the associated marking on $\mathbb{F}(\mathbb{C})^{\dagger} = (\mathbb{F}(\mathbb{C}), E_{\mathbb{F}(\mathbb{C})^{\dagger}}, T_{\mathbb{F}(\mathbb{C})^{\dagger}})$. Then the induced morphism

$$(\mathbb{C}, E_{\mathbb{C}^\dagger}, T_{\mathbb{C}^\dagger} \subset \sharp) \longrightarrow (\mathbb{F}(\mathbb{C}), E_{\mathbb{F}(\mathbb{C})^\dagger}, T_{\mathbb{F}(\mathbb{C})^\dagger} \subset \sharp)$$

 $is \ MB$ -anodyne.

Proof. Let us denote $\mathbb{C}^{\dagger}_{\sharp} = (\mathbb{C}, E_{\mathbb{C}^{\dagger}}, T_{\mathbb{C}^{\dagger}} \subseteq \sharp)$ and similarly $\mathbb{F}(\mathbb{C})^{\dagger}_{\sharp}$. We consider a pushout diagram over \mathbb{D}

$$\begin{array}{cccc} \mathbb{C}^{\dagger} & \longrightarrow & \mathbb{C}^{\dagger}_{\sharp} \\ & & & \downarrow \simeq \\ & & \downarrow \simeq \\ \mathbb{F}(\mathbb{C})^{\dagger} & \longrightarrow & \mathbb{F}(\mathbb{C})^{\dagger}_{\diamond} \end{array}$$

whose vertical morphisms are all weak equivalences. We will show that the induced map $s : \mathbb{F}(\mathbb{C})^{\dagger}_{\diamond} \to \mathbb{F}(\mathbb{C})^{\dagger}_{\sharp}$ is anodyne. Let $\sigma : \Delta^2 \to \mathbb{F}(\mathbb{C})^{\dagger}_{\diamond}$ be a triangle. Note that by construction $E_0^*(\sigma)$ is fully lean scaled. This implies that $E_0^*(\sigma)$ has all faces lean except possibly the face missing the vertex 1. Since the triangle $\{0,1,2\}$ is thin we can take a pushout along an **MB**-anodyne morphism to lean scaled the face missing 1.

Now we consider $E_1^*(\sigma)$ and observe that the face missing 0 and 3 are lean. Additionally we see

that $d_1(E_1^*(\sigma)) = d_1(E_0^*(\sigma))$ so it follows that all faces are already lean except the face missing the vertex 2. We scale the aforementioned face after noting that $\{1,2,3\}$ is a thin triangle. A similar argument then shows that all of the faces of $E_2^*(\sigma)$ are scaled except possible $d_3(E_2^*(\sigma)) = \sigma$. However since the last vertex is marked and the triangle $\{0,2,3\}$ is scaled the result follows. \square

Definition 4.23. Let $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ be a functor of marked ∞ -bicategories. Given $d \in \mathbb{D}$ we denote by $\mathbb{C}_{d^{\gamma}}^{\dagger}$ the fibre over the object d of the morphism $\mathbb{F}(\mathbb{C})^{\dagger} \to \mathbb{D}$.

Definition 4.24. Let $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ be a functor of marked ∞ -bicategories. We define the 2-Cartesian fibration $\mathbb{C}^{\dagger}_{\mathbb{D}^{\prime}} \to \mathbb{D}$ to be a fibrant replacement of the object $(\mathbb{F}(\mathbb{C}), E_{\mathbb{F}(\mathbb{C})^{\dagger}}, T_{\mathbb{F}(\mathbb{C})^{\dagger}} \subset \sharp)$ in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\mathbb{D}}$.

Proposition 4.25. Let $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ be a functor of marked ∞ -bicategories. Then the following statements are equivalent:

- i) The map f is marked cofinal.
- ii) The morphism $(\mathbb{F}(\mathbb{C}), E_{\mathbb{F}(\mathbb{C})^{\dagger}}, T_{\mathbb{F}(\mathbb{C})^{\dagger}} \subset \sharp) \to (\mathbb{F}(\mathbb{D}), E_{\mathbb{F}(\mathbb{D})^{\dagger}}, T_{\mathbb{F}(\mathbb{D})^{\dagger}} \subset \sharp)$ is a weak equivalence.
- iii) For every $d \in \mathbb{D}$ we have an equivalence of ∞ -categorical localizations $L_W(\mathbb{C}_{d^{\mathcal{T}}}^{\dagger}) \to L_W(\mathbb{D}_{d^{\mathcal{T}}}^{\dagger})$.

Proof. The equivalence i) \iff ii) follows from Lemma 4.22 and the functoriality of the free fibration. To finish the proof we will show that ii) \iff iii).

Consider projective-fibrant functors $\mathcal{F}_{\mathbb{C}}, \mathcal{F}_{\mathbb{D}} : \mathfrak{C}^{\mathrm{sc}}[\mathbb{D}]^{\mathrm{op}} \to \mathrm{Set}_{\Delta}^{\mathbf{ms}}$ equipped with equivalences $\mathrm{Un}_{\mathbb{D}}(\mathcal{F}_{\mathbb{C}}) \simeq \mathbb{F}(\mathbb{C})$ and $\mathrm{Un}_{\mathbb{D}}(\mathcal{F}_{\mathbb{D}}) \simeq \mathbb{F}(\mathbb{D})$. We can define new functors $\mathcal{F}_{\mathbb{C}}^{\dagger}$ and $\mathcal{F}_{\mathbb{D}}^{\dagger}$ and a morphism $\mathcal{F}_{\mathbb{C}}^{\dagger} \to \mathcal{F}_{\mathbb{D}}^{\dagger}$ via pushout, e.g.,

$$\begin{array}{ccc} \mathbb{S}\mathrm{t}_D(\mathbb{F}(\mathbb{C})) & \stackrel{\sim}{\longrightarrow} & \mathcal{F}_{\mathbb{C}} \\ & & \downarrow & & \downarrow \\ \\ \mathbb{S}\mathrm{t}_D(\mathbb{F}(\mathbb{C})^\dagger) & \stackrel{\sim}{\longrightarrow} & \mathcal{F}_{\mathbb{C}}^\dagger \end{array}$$

We thus see that the induced map on fibrant reparlements $\mathsf{R}(\mathbb{U}\mathsf{n}_{\mathbb{D}}(\mathcal{F}^{\dagger}_{\mathbb{C}})) \to \mathsf{R}(\mathbb{U}\mathsf{n}_{\mathbb{D}}(\mathcal{F}^{\dagger}_{\mathbb{C}}))$ is a model for $\mathbb{C}^{\dagger}_{\mathbb{D}^{\uparrow}} \to \mathbb{D}^{\dagger}_{\mathbb{D}^{\uparrow}}$. Moreover, the pushout is computed pointwise. Unravelling the definition, we note that for each $d \in \mathbb{D}$ the square

$$\begin{array}{ccc} \operatorname{St}_*(\mathbb{C}_{d^{\wedge}}) & \stackrel{\sim}{\longrightarrow} & \operatorname{St}_{\mathbb{D}}(\mathbb{F}(\mathbb{C}))(d) \\ \downarrow & & \downarrow \\ & \operatorname{St}_*(\mathbb{C}_{d^{\wedge}}^{\dagger}) & \stackrel{\sim}{\longrightarrow} & \operatorname{St}_{\mathbb{D}}(\mathbb{F}(\mathbb{C})^{\dagger})(d) \end{array}$$

is homotopy pushout. We thus have natural equivalences

$$\mathbb{C}_{d^{\wedge}}^{\dagger} \simeq \mathbb{S} \mathrm{t}_{*}(\mathbb{C}_{d^{\wedge}}^{\dagger}) \simeq \mathbb{S} \mathrm{t}_{\mathbb{D}}(\mathbb{F}(\mathbb{C})^{\dagger})(d) \simeq \mathcal{F}_{\mathbb{C}}^{\dagger}(d).$$

Finally, we note that there are canonical natural identifications

$$\mathbb{C}_{\mathbb{D}^{\uparrow}}^{\dagger} \times_{\mathbb{D}} \{d\} \simeq \mathsf{R}(\mathcal{F}_{\mathbb{C}}^{\dagger})(d) \simeq L_{W}(\mathcal{F}_{\mathbb{C}}^{\dagger}(d))$$

so that we get a commutative diagram

$$\mathbb{C}_{\mathbb{D}^{\uparrow}}^{\dagger} \times_{\mathbb{D}} \{d\} \xrightarrow{\simeq} L_{W}(\mathbb{C}_{d^{\uparrow}}^{\dagger})$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathbb{D}_{\mathbb{D}^{\uparrow}}^{\dagger} \times_{\mathbb{D}} \{d\} \xrightarrow{\simeq} L_{W}(\mathbb{D}_{d^{\uparrow}}^{\dagger})$$

The proposition then follows from [4, Prop. 4.25].

Proposition 4.26. Let $p: \mathbb{X} \to \mathbb{D}$ be a 2-Cartesian fibration such that every triangle in \mathbb{X} is lean. Suppose that for every $d \in \mathbb{D}$ there exists an initial object $i_d \in \mathbb{X}_d$ in the fibre over d. Then the restriction of p to the the marked biscaled simplicial set spanned by initial objects $\hat{p}: \hat{\mathbb{X}} \to \mathbb{D}$ is a trivial fibration of scaled simplicial sets.

Proof. We first show that \hat{p} is a fibration in the model structure on $\operatorname{Set}_{\Delta}^{\mathbf{sc}}$. Since p is a 2-Cartesian fibration, it is easy to see that \hat{p} has the right lifting property against all scaled anodyne morphisms. By virtue of [10, Cor 6.4] it will suffice to check that \hat{p} is an isofibration. Let $d_0 \to d_1 = p(x_1)$ be an equivalence in \mathbb{D} and pick a lift $x_0 \to x_1$ such that x_1 is initial in the fibre over d_1 . Let us pick an initial object \hat{x}_0 and consider the composite morphism $u: \hat{x}_0 \to x_1$. We claim that u is an equivalence. Let $\mathbb{D} \subset \mathbb{D}$ denote the underlying ∞ -category of \mathbb{D} and let us consider a pullback diagram

$$X_{\mathbb{D}} \longrightarrow X$$

$$\downarrow \qquad \qquad \downarrow^{p}$$

$$\mathcal{D} \longrightarrow \mathbb{D}$$

where the left-most vertical morphism is a Cartesian fibration of ∞ -categories. Let $\hat{\mathbf{X}}_{\mathcal{D}}$ denote the restriction to the full subcategory on fibrewise initial objects. Then it follows from [21, Prop. 2.4.4.9] that the restriction $\hat{\mathbf{X}}_{\mathcal{D}} \to \mathcal{D}$ is a trivial Kan fibration. In particular it detects equivalences and the claim follows.

We have thus reduced our problem to showing that \hat{p} is a bicategorical equivalence. By our hypothesis it follows that \hat{p} is surjective on objects. To finish the proof we will check that for every pair of objects $x, y \in \mathbf{X}$ the induced morphism of mapping ∞ -categories

$$\hat{p}_{x,y} \colon \operatorname{Map}_{\hat{\mathbf{X}}}(x,y) \longrightarrow \operatorname{Map}_{\mathbb{D}}(\hat{p}(x),\hat{p}(y))$$

is an equivalence. Note that since every 2-simplex in X is lean it follows that not only is $p_{x,y}$ a coCartesian fibration, it is also a left fibration. Therefore we reduce our problem to showing that the fibres of $\hat{p}_{x,y}$ are all contractible. This follows from our hypothesis using [4, Proposition 4.21]

Lemma 4.27. Let \mathbb{D}^{\dagger} be a marked ∞ -bicategory. Then the 2-Cartesian fibration $\mathbb{D}_{\mathbb{D}^{\uparrow}}^{\dagger} \to \mathbb{D}$ satisfies the hypothesis of Proposition 4.26.

Proof. Recall the model for $\mathbb{D}_{\mathbb{D}^{\uparrow}}^{\dagger}$ given in Proposition 4.25. As a direct consequence we observe that every triangle in $\mathbb{D}_{\mathbb{D}^{\uparrow}}^{\dagger}$ is lean. We claim that for every $d \in \mathbb{D}$ the identity morphism id_d on d is initial in its corresponding fibre. Note that we can identify the fibre over d with $L_W(\mathbb{D}_{d^{\uparrow}}^{\dagger})$. Since for every object $f: d \to d'$, the mapping ∞ -category $\mathrm{Map}_{\mathbb{D}_{d^{\uparrow}}}(\mathrm{id}_d, f)$ is contractible due to Lemma 4.28 it follows that id_d is initial in the localisation.

Lemma 4.28. Let \mathbb{D} be an ∞ -bicategory. Let $\mathrm{id}_d: d \to d$ and $e: d \to d'$ be a pair of edges in \mathbb{D} such that id_d is degenerate. Let $r: \Delta^1 \times \Delta^1 \to \Delta^1$ be the morphism that sends every vertex to 0

except (1,1) which gets sent to 1. Then the composite

$$\eta_e \colon \Delta^1 \times \Delta^1 \xrightarrow{r} \Delta^1 \xrightarrow{e} \mathbb{D}$$

defines a terminal object in the mapping ∞ -category $\operatorname{Map}_{\mathbb{D}_{d^*}}(\operatorname{id}_d, f)$.

Proof. We will show that every boundary $\partial \alpha : \partial \Delta^n \to \operatorname{Map}_{\mathbb{D}_{d_f^n}}(\operatorname{id}_d, f)$ such that $\partial \alpha(n) = \eta_e$ can be extended to an n-simplex $\alpha : \Delta^n \to \operatorname{Map}_{\mathbb{D}_{d_f^n}}(\operatorname{id}_d, f)$.

We define a subsimplicial subset (with the inherited scaling) $S^{n+1} \subset \Delta^1 \otimes \Delta^{n+1}$ consisting of precisely those simplices σ satisfying at least one of the conditions below:

- The simplex σ is contained in $\Delta^{\{0\}} \times \Delta^{n+1}$.
- Given $j \in [n+1]$ the simplex σ skips vertices of the form (ε, j) with $\varepsilon \in \{0, 1\}$.

Unraveling the definitions we see that we need to solve the associated lifting problem

$$\begin{array}{ccc} \mathcal{S}^{n+1} & \stackrel{\partial \alpha}{\longrightarrow} & \mathbb{D} \\ \downarrow^{\iota} & & & \\ \Delta^1 \otimes \Delta^{n+1} & & & \end{array}$$

We will abuse notation and denote by $\Delta^1 \otimes \Delta^{n+1}$ the Gray product where we are additionally scaling the triangles $(0,j) \to (0,j+1) \to (1,j+1)$ whenever j < n and the triangle $(n,0) \to (n+1,0) \to (1,n+1,1)$. We will carry this additional scaling to \mathcal{S}^{n+1} . Note that by construction $\partial \alpha$ sends those triangles to thin simplices in \mathbb{D} . We produce a factorization

$$S^{n+1} \xrightarrow{u} \mathcal{R}^{n+1} \xrightarrow{v} \Delta^1 \otimes \Delta^{n+1}$$

where \mathbb{R}^{n+1} consists of those simplices of $\Delta^1 \otimes \Delta^{n+1}$ that skip the vertex (1,1). It is easy to see that u is scaled anodyne and that v fits into a pushout square

$$\Lambda_0^{n+1} \coprod_{\Delta^{\{0,1\}}} \Delta^0 \longrightarrow \Delta^{n+1} \coprod_{\Delta^{\{0,1\}}} \Delta^0$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathcal{R}^{n+1} \xrightarrow{v} \Delta^1 \otimes \Delta^{n+1}$$

since the triangle $\{0,1,n\}$ is thin by construction it follows that v is also scaled anodyne. The result now follows.

We now arrive at the main theorem of this section, which provides a computational criterion for cofinality.

Theorem 4.29. Let $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ be a functor of marked ∞ -bicategories. Then the following statements are equivalent

- 1. The functor f is marked cofinal.
- 2. For every $d \in \mathbb{D}$ the functor f induces an equivalence of ∞ -categorical localizations $L_W(\mathbb{C}_{d\uparrow}^{\dagger}) \to L_W(\mathbb{D}_{d\uparrow}^{\dagger})$.
- 3. The following conditions hold:
 - i) For every $d \in \mathbb{D}$ there exists a morphism $g_d : d \to f(c)$ which is initial in $L_W(\mathbb{C}_{d\uparrow}^{\dagger})$ and $L_W(\mathbb{D}_{d\uparrow}^{\dagger})$.

- ii) Every marked morphism $d \to f(c)$ defines an initial object in $L_W(\mathbb{C}_{d^{\uparrow}}^{\dagger})$.
- iii) For any marked morphism $d \to b$ in \mathbb{D} the induced functor $L_W(\mathbb{C}_{b\nearrow}^{\dagger}) \to L_W(\mathbb{C}_{d\nearrow}^{\dagger})$ preserves initial objects.

Proof. By Proposition 4.25 it will suffice to show that 2 holds if and only if 3 holds. Let us suppose that 2 holds. Since by hypothesis the morphism $L_W(\mathbb{C}_{d^{\gamma}}^{\dagger}) \to L_W(\mathbb{D}_{d^{\gamma}}^{\dagger})$ is an equivalence of ∞ -categories we can pick an object $d \to f(c)$ whose image in $L_W(\mathbb{D}_{d^{\gamma}}^{\dagger})$ is equivalent to id_d . Since equivalences preserve and detect initial objects we see that condition i is satisfied. To see that condition i holds we just note that every marked morphism in $L_W(\mathbb{D}_{d^{\gamma}}^{\dagger})$ is equivalent to id_d . Using again that equivalences detect initial objects the claim follows. For the final condition we consider a commutative diagram

$$L_{W}(\mathbb{C}_{b\nearrow}^{\dagger}) \xrightarrow{\simeq} L_{W}(\mathbb{D}_{b\nearrow}^{\dagger})$$

$$\downarrow \qquad \qquad \downarrow$$

$$L_{W}(\mathbb{C}_{d\nearrow}^{\dagger}) \xrightarrow{\simeq} L_{W}(\mathbb{D}_{d\nearrow}^{\dagger})$$

It is now clear that condition iii) holds if the right-most vertical morphism preserves initial objects. We observe that this map sends the identity on b to an object represented by a marked morphism and thus preserves initial objects.

Now let us suppose that the conditions in 3 are satisfied. Using Proposition 4.25 we see that it will suffice to show that the induced morphism of fibrant replacements $\mathcal{A}_f: \mathbb{C}_{\mathbb{D}^{\uparrow}}^{\dagger} \to \mathbb{D}_{\mathbb{D}^{\uparrow}}^{\dagger}$ is an equivalence of 2-Cartesian fibrations. Notice that by assumption it follows that $\mathbb{C}_{\mathbb{D}^{\uparrow}}^{\dagger}$ satisfies the hypothesis of Proposition 4.26. Let us denote by $\hat{\mathbb{C}}_{\mathbb{D}^{\uparrow}}^{\dagger}$ the full marked-biscaled simplicial set spanned by fibrewise initial objects and similarly for $\hat{\mathbb{D}}_{\mathbb{D}^{\uparrow}}^{\dagger}$. Observe that due to Proposition 4.26 we have a section

$$s_f \colon \mathbb{D}_{\sharp} \longrightarrow \hat{\mathbb{C}}_{\mathbb{D}_{f}^{\uparrow}}^{\dagger} \longrightarrow \mathbb{C}_{\mathbb{D}_{f}^{\uparrow}}^{\dagger} \text{ where } \mathbb{D}_{\sharp} = (\mathbb{D}, E_{\mathbb{D}}, T_{\mathbb{D}} \subset \sharp).$$

We can pick the section so that each d gets sent to $g_d: d \to f(c)$ as in condition i). We claim that s_f sends marked edges in $\mathbb{D}_{\sharp}^{\dagger} = (\mathbb{D}, E_{\mathbb{D}^{\dagger}}, T_{\mathbb{D}^{\dagger}} \subset \sharp)$ to Cartesian edges in $\mathbb{C}_{\mathbb{D}^{\uparrow}}^{\dagger}$. Let $e: d \to b$ be a marked edge in $\mathbb{D}_{\sharp}^{\dagger}$ and pick a Cartesian lift of e, $\hat{e}: \Delta^1 \to \mathbb{C}_{\mathbb{D}^{\uparrow}}^{\dagger}$ such that $\hat{e}(1) = g_b$. By condition iii), we have that $\hat{e}(0)$ is initial in the fibre over d. We consider the commutative diagram

$$\begin{array}{ccc} \Lambda_2^2 \stackrel{\sigma}{\longrightarrow} \mathbb{C}_{\mathbb{D}^{\wedge}}^{\dagger} \\ \downarrow & \stackrel{\theta}{\downarrow} & \downarrow \\ \Delta^2 \stackrel{s_0(e)}{\longrightarrow} \mathbb{D} \end{array}$$

with $\sigma(1 \to 2) = \hat{e}$ and $\sigma(0 \to 2) = s_f(e)$. The triangle θ is thin by construction and the edge $0 \to 1$ is an equivalence since it is a morphism between initial objects. It follows that $s_f(e)$ is Cartesian. We can now use Lemma 4.22 to produce a solution to the lifting problem

$$\begin{array}{ccc} \mathbb{D}_{\sharp}^{\dagger} & \stackrel{s_f}{\longrightarrow} \mathbb{C}_{\mathbb{D}_{\ell}^{\prime}}^{\dagger} \\ \downarrow^{i_{\mathbb{D}}} & \mathcal{I}_f & \\ \mathbb{D}_{\mathbb{D}_{\ell}^{\prime}}^{\dagger} & & \end{array}$$

We claim that \mathcal{A}_f and \mathcal{I}_f are mutually inverse. First we observe that $\mathcal{A}_f \circ s_f$ is a section of $\mathbb{D}_{\mathbb{D}_f}^{\dagger}$ that maps each object $d \in \mathbb{D}$ to an initial object in the fibre. Using the fact that $\hat{\mathbb{D}}_{\mathbb{D}_f}^{\dagger} \to \mathbb{D}$ is a trivial fibration we can construct a homotopy over \mathbb{D} ,

$$H_{\mathbb{D}} \colon \Delta^1 \times \mathbb{D}_{\sharp} \longrightarrow \mathbb{D}_{\mathbb{D}^{7}}^{\dagger}$$

between $i_{\mathbb{D}}$ and $\mathcal{A}_f \circ s_f$. Observe that $i_{\mathbb{D}}(d) = \mathrm{id}_d$ so it maps every object to an initial object in the fibre. By construction the components of $H_{\mathbb{D}}$ are morphisms between initial objects and thus equivalences. Let $e: d \to b$ be a marked morphism in $\mathbb{D}_{\sharp}^{\dagger}$ then it follows that $H_{\mathbb{D}}(0 \to 1, e)$ is marked in $\mathbb{D}_{\mathbb{D}_{\uparrow}}^{\dagger}$. We can therefore upgrade the homotopy $H_{\mathbb{D}}$ to a marked homotopy $H_{\mathbb{D}}: (\Delta^1)^{\sharp} \times \mathbb{D}_{\sharp}^{\dagger} \to \mathbb{D}_{\mathbb{D}_{\uparrow}}^{\dagger}$. To see that $\mathcal{A}_f \circ \mathcal{I}_f \simeq \mathrm{id}$ it suffices to check that $\mathcal{A}_f \circ \mathcal{I}_f \circ i_{\mathbb{D}} \simeq i_{\mathbb{D}}$, however we have

$$\mathcal{A}_f \circ \mathcal{I}_f \circ i_{\mathbb{D}} = \mathcal{A}_f \circ s_f \simeq i_{\mathbb{D}}.$$

Let us fix some notation $i_f: \mathbb{C}_{\sharp}^{\dagger} \to \mathbb{C}_{\mathbb{D}_{\uparrow}^{\uparrow}}^{\dagger}$ and $i_{\mathbb{C}}: \mathbb{C}_{\sharp}^{\dagger} \to \mathbb{C}_{\mathbb{C}_{\uparrow}^{\uparrow}}^{\dagger}$. In order to show that $\mathcal{I}_f \circ \mathcal{A}_f \simeq \mathrm{id}$ we will show that $\mathcal{I}_f \circ \mathcal{A}_f \circ i_f \simeq i_f$. Consider the following pullback square

$$f^* \left(\mathbb{C}_{\mathbb{D} \uparrow}^{\dagger} \right) \xrightarrow{\phi} \mathbb{C}_{\mathbb{D} \uparrow}^{\dagger}$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathbb{C} \xrightarrow{f} \mathbb{D}$$

and note that $i_f = \phi \circ \mathcal{B}_f \circ i_{\mathbb{C}}$ where $\mathcal{B}_f : \mathbb{C}_{\mathbb{C}^{\uparrow}}^{\dagger} \to f^* \left(\mathbb{C}_{\mathbb{D}^{\uparrow}}^{\dagger} \right)$ is the obvious morphism. We now observe that $\mathcal{I}_f \circ \mathcal{A}_f \circ \phi = \phi \circ f^*(\mathcal{I}_f) \circ f^*(\mathcal{A}_f)$. A similar argument as before shows that

$$f^*(\mathcal{I}_f) \circ f^*(\mathcal{A}_f) \circ \mathcal{B}_f \circ i_{\mathbb{C}} \simeq \mathcal{B}_f \circ i_{\mathbb{C}}.$$

This is due to the fact that both sides of the equation describe sections of $f^*\left(\mathbb{C}_{\mathbb{D}^{\uparrow}}^{\dagger}\right)$ with values in initial objects. Note that $\mathcal{B}_f \circ i_{\mathbb{C}}(c)$ is initial as a consequence of condition ii). We conclude the proof by finally noting

$$\mathcal{I}_f \circ \mathcal{A}_f \circ i_f = \mathcal{I}_f \circ \mathcal{A}_f \circ \phi \circ \mathcal{B}_f \circ i_{\mathbb{C}} = \phi \circ f^*(\mathcal{I}_f) \circ f^*(\mathcal{A}_f) \circ \mathcal{B}_f \circ i_{\mathbb{C}} \simeq \phi \circ \mathcal{B}_f \circ i_{\mathbb{C}} = i_f$$

We have shown that $\mathcal{I}_f \circ \mathcal{A}_f \simeq \mathrm{id}$ and the theorem now follows.

We finish this section by studying the case where $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ is a functor between strict 2-categories equipped with a marking. For the rest of the section we will denote $\mathbb{C}^{\dagger} = \mathrm{N}^{\mathrm{sc}}(\mathbb{C})^{\dagger}$ (resp. $\mathbb{D}^{\dagger} := \mathrm{N}^{\mathrm{sc}}(\mathbb{D})^{\dagger}$) where the marking comes from the marking in \mathbb{C}^{\dagger} (resp. \mathbb{D}^{\dagger}). Our goal is to relate (nerves of) the comma 2-categories of Definition 2.2 with the fibres of the free 2-Cartesian fibration thus simplifying the conditions of Theorem 4.29.

Definition 4.30. Let $f: \mathbb{C} \to \mathbb{D}$ be a functor of strict 2-categories. We define a new 2-category $Fr(\mathbb{C})$ as follows:

- Objects are given by morphisms $u: d_0 \to f(c_0)$ where $d_0 \in \mathbb{D}$ and $c_0 \in \mathbb{C}$.
- A morphism $\varphi_0: u \to v$ from $u: d_0 \to f(c_0)$ to $v: d_1 \to f(c_1)$ is given by a pair of morphisms $a_0: d_0 \to d_1$ and $\alpha_0: c_0 \to c_1$ and a 2-morphism $\theta_{\varphi_0}: f(\alpha) \circ u \Rightarrow v \circ a$.
- A 2-morphism $\varepsilon: \varphi_0 \to \varphi_1$ is given by a pair of 2-morphisms $\psi: a_0 \Rightarrow a_1$ and $\zeta: \alpha_0 \Rightarrow \alpha_1$

such that the followign diagram commutes

$$f(\alpha_0) \circ u \xrightarrow{f(\zeta) * u} f(\alpha_1) \circ u$$

$$\downarrow \theta_{\varphi_0} \qquad \qquad \downarrow \theta_{\varphi_1}$$

$$v \circ a_0 \xrightarrow{v * \psi} v \circ a_1$$

There is an obvious 2-functor $Fr(\mathbb{C}) \to \mathbb{D}$ which is easily verified to be a 2-Cartesian fibration. In particular one observes the following:

- A morphism in $Fr(\mathbb{C})$ is Cartesian if the associated morphism $\alpha: c_0 \to c_1$ is an equivalence in \mathbb{C} and the 2-morphism φ_0 is invertible.
- A 2-morphism in $Fr(\mathbb{C})$ is coCartesian if the associated 2-morphism $\zeta: \alpha_0 \Rightarrow \alpha_1$ is invertible.

One immediately sees that the fibres of $Fr(\mathbb{C})$ are precisely the categories $\mathbb{C}_{d\nearrow}$ of Definition 2.2.

Remark 4.31. As a direct consequence of [4, Theorem 4.29] we see that the induced morphism $N^{sc}(Fr(\mathbb{C})) \to \mathbb{D}$ is a 2-Cartesian fibration. We further observe that there is an strict 2-functor $\mathbb{C} \to Fr(\mathbb{C})$. We will see at the end of the section that $Fr(\mathbb{C})$ is another model for the free 2-Cartesian fibration on the functor f.

Remark 4.32. Suppose we are given a morphism of marked strict 2-categories $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$. Then we can construct a marked 2-category $Fr(\mathbb{C})^{\dagger}$ by declaring an edge in $Fr(\mathbb{C})$ to be marked if and only if it is Cartesian or the associated morphism $\alpha: c_0 \to c_1$ is marked in \mathbb{C}^{\dagger} and the 2-morphism φ_0 is invertible. We denote by $N^{sc}(Fr(\mathbb{C}))^{\dagger}$ the associated **MB** simplicial set.

Before we continue, we must provide a good characterization of the simplices of $N^{sc}(Fr(\mathbb{C}))$. As it turns out, we can view $N^{sc}(Fr(\mathbb{C}))$ as a simplicial subset of $\mathbb{F}(\mathbb{C})$, and we will use this to provide an alternate characterization of the simplices of the former. To this end we fix some terminology. Let us call a 2-simplex of $\Delta^1 \widehat{\otimes} \Delta^n$ contrary if it is scaled.

An *n*-simplex σ of $\mathbb{F}(\mathbb{C})$ consists of a commutative diagram

$$\begin{array}{ccc} \Delta^1 \widehat{\otimes} \Delta^n & \stackrel{\phi^{\sigma}}{\longrightarrow} & \mathbb{D} \\ & & & \uparrow^f \\ \{1\} \times \Delta^n & \stackrel{\rho^{\sigma}}{\longrightarrow} & \mathbb{C} \end{array}$$

We will call such a simplex tame if it sends contrary 2-simplices to identities.⁵ By construction, the tame simplices form a simplicial subset of $\mathbb{F}(\mathbb{C})$, which we will denote by $\mathrm{Tame}(\mathbb{C}, \mathbb{D})$. When this is equipped with the marking and biscaling induced by $\mathbb{F}(\mathbb{C})$, we denote it by $\mathrm{Tame}(\mathbb{C}, \mathbb{D})^{\dagger}$.

Lemma 4.33. There is an isomorphism

$$\operatorname{Tame}(\mathbb{C}, \mathbb{D})^{\dagger} \cong \operatorname{N}^{\operatorname{sc}}(\operatorname{Fr}(\mathbb{C}))^{\dagger}$$

of marked-biscaled simplicial sets.

Proof. We will prove that the underlying simplicial set $Tame(\mathbb{C}, \mathbb{D})$ is 3-coskeletal, reducing the proof to a straightforward check on 3-truncations.

⁵Notice that this definition is only sensible because $\mathbb{D} := N^{sc}(\mathbb{D})$. Otherwise, there is no good notion of identity 2-simplices which are neither left nor right degenerate.

Consider a morphism $\partial \Delta^n \to \text{Tame}(\mathbb{C}, \mathbb{D})$ where n > 3. This corresponds to a diagram

$$\begin{array}{ccc}
\Delta^1 \times \partial \Delta^n & \stackrel{\phi}{\longrightarrow} & \mathbb{D} \\
\uparrow & & \uparrow f \\
\{1\} \times \partial \Delta^n & \stackrel{\rho}{\longrightarrow} & \mathbb{C}
\end{array}$$

We now note that \mathbb{C} and \mathbb{D} are, themselves 3-coskeletal, and thus, in particular, they admit unique horn fillers for all horns of dimension 5 or higher. Using, e.g., the filtration of [21, Prop. 2.1.2.6], we see that ϕ has a unique extension to a map $\Delta^1 \times \Delta^n \to \mathbb{D}$. Since ρ clearly has a unique extension to a map $\Delta^n \to \mathbb{C}$, we can obtain an extension

$$\begin{array}{ccc}
\Delta^{1} \times \Delta^{n} & \xrightarrow{\widetilde{\phi}} & \mathbb{D} \\
\uparrow & & \uparrow f \\
\{1\} \times \Delta^{n} & \xrightarrow{\widetilde{g}} & \mathbb{C}
\end{array} \tag{*}$$

of the diagram above.

Moreover, since n > 3, every 2-simplex of $\Delta^1 \times \Delta^n$ is contained in $\Delta^1 \times \partial \Delta^n$. Consequently, the fact that ϕ arises from a map $\partial \Delta^n \to \operatorname{Tame}(\mathbb{C}, \mathbb{D})$ implies that the diagram (*) defines an n-simplex in $\operatorname{Tame}(\mathbb{C}, \mathbb{D})$.

The remaining low-dimensional checks are left to the reader.

Remark 4.34. Note that the argument above can in fact be repurposed to show that $\mathbb{F}(\mathbb{C})$ is itself 3-coskeletal in our present setting. However, in spite of their equivalence, $\mathbb{F}(\mathbb{C})$ will not be isomorphic to $N^{sc}(Fr(\mathbb{C}))$, as the former has significantly more 1- and 2-simplices.

Remark 4.35. By construction, the canonical morphism $\mathbb{C} \to \mathbb{F}(\mathbb{C})$ factors through Tame(\mathbb{C}, \mathbb{D}).

Lemma 4.36. Let $\sigma: \Delta^n \to \mathrm{Tame}(\mathbb{C}, \mathbb{D})$ be an n-simplex. Then each extension $E_j^*(\sigma): \Delta^{n+1} \to \mathbb{F}(\mathbb{C})$ factors through $\mathrm{Tame}(\mathbb{C}, \mathbb{D})$.

Proof. This follows immediately from unraveling the definitions.

Definition 4.37. We denote by $\operatorname{Tame}(\mathbb{C}, \mathbb{D})^{\natural}$ the marking and biscaling induced by $\mathbb{F}(\mathbb{C})^{\natural}$. Similarly, we denote by $\operatorname{Tame}(\mathbb{C}, \mathbb{D})^{\dagger}$ the marking and biscaling induced by $\mathbb{F}(\mathbb{C})^{\dagger}$.

Proposition 4.38. The morphism $\mathbb{C}^{\natural} \to \operatorname{Tame}(\mathbb{C}, \mathbb{D})^{\natural}$ is MB-anodyne over \mathbb{D} .

Proof. This is identical to the proof of Theorem 4.15 once we redefine Z_n to consist of n-simplices of Tame(\mathbb{C}, \mathbb{D}), together with j-extensions of these simplices.

Theorem 4.39. Let $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ be a functor between marked strict 2-categories, and let $f: \mathbb{C}^{\dagger} \to \mathbb{D}^{\dagger}$ denote the induced morphism of MS simplicial sets. Then the following hold:

• There exists a commutative diagram over $\mathbb D$

$$\begin{array}{ccc} \mathbb{C}^{\dagger} & \longrightarrow & \mathrm{N^{sc}}(\mathrm{Fr}(\mathbb{C})) \\ \downarrow & & \searrow \\ & & & & \\ \mathbb{F}(\mathbb{C})^{\dagger} & & & \end{array}$$

such that each morphism in the diagram is a 2-Cartesian equivalence.

• For every $d \in \mathbb{D}$ the map Ξ induces an equivalence of MS simplicial sets $\mathbb{C}_{d^{\uparrow}}^{\dagger} \xrightarrow{\cong} N^{\mathrm{sc}}(\mathbb{C}_{d^{\uparrow}}^{\dagger})$.

Proof. First let us assume that the marking on both $\mathbb{C}^{\dagger} = \mathbb{C}^{\natural}$ and $\mathbb{D}^{\dagger} = \mathbb{D}^{\dagger}$ only consists of equivalences so that the marking on both $\mathbb{F}(\mathbb{C})$ and $N^{sc}(Fr(\mathbb{C})^{\natural})$ is precisely given by Cartesian edges. Recall the filtration defined in Theorem 4.15. First we will define a morphism

Since $Z_1 \to \mathbb{F}(\mathbb{C})$ is **MB**-anodyne and $N^{sc}(Fr(\mathbb{C}))$ is a 2-Cartesian fibration, we can pick an extension of Ξ_1 to the desired Ξ . Observe that we can map the objects of Z_1 isomorphically to those of $N^{sc}(Fr(\mathbb{C}))$. Given $e: \Delta^1 \to Z_1$ we see that this data precisely amounts to morphisms $u_i: d_i \to f(c_i)$ for $i = 0, 1, a: d_0 \to d_1, \alpha: c_0 \to c_1$ and $g: d_0 \to f(c_1)$ together with a pair of 2-morphisms $\varepsilon: g \xrightarrow{\cong} f(\alpha) \circ u_0$ and $\theta: g \Rightarrow u_1 \circ a$ such that ε is invertible. We can then map e to an edge $\Xi_1(e)$ defined by the same 1-morphisms but with associated 2-morphism $\theta \circ \varepsilon^{-1}$. One perfoms a similar construction for mapping the non-degenerate 2-simplices contained in Z_1 thus giving a definition for Ξ_1 .

We can now observe that in the case that \mathbb{C}^{\dagger} comes equipped with a general marking (containing the equivalences) we have a homotopy pushout $(\operatorname{Set}_{\Lambda}^{\mathbf{mb}})_{/\mathbb{D}}$

$$\begin{array}{ccc} \mathbb{F}(\mathbb{C})^{\natural} & \longrightarrow & N^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C})^{\natural}) \\ \downarrow & & \downarrow \\ \mathbb{F}(\mathbb{C})^{\dagger} & \longrightarrow & N^{\mathrm{sc}}(\mathrm{Fr}(\mathbb{C})^{\dagger}) \end{array}$$

This shows it will suffice to prove the case where only the equivalences are marked. This follows from 2-out-of-3 after noting that $\mathbb{C}^{\natural} \to N^{sc}(Fr(\mathbb{C})^{\natural})$ is an equivalence. This follows immediately from Proposition 4.38.

Remark 4.40. The significance of this result is twofold. Most importantly, it shows that, when considering diagrams indexed over strict 2-categories, the criteria for marked cofinality can be expressed in terms of the strict slice 2-categories. Consequently, the criteria for cofinality become much easier to explicitly check in this case.

Of lesser significance, but still of interest, is the second consequence. Since we can identify $N^{sc}(\mathbb{C}_{d\uparrow})$ and $\mathbb{C}_{d\uparrow}$, the criteria of Theorem 4.29 precisely agree with those of [2, Thm 4.0.1]. Theorem 4.29 thus generalizes [2, Thm 4.0.1], as expected.

A. The Relative 2-Nerve

There is a special case of most ∞ -categorical Grothendieck constructions in which the computation of the right adjoints can be greatly simplified. When the base is suitably strict, it is possible to define a *relative nerve*, which computes the Grothendieck construction of a functor. The aim of this appendix is to provide a relative nerve construction which takes as input a $\operatorname{Set}_{\Lambda}^+$ -enriched functor

$$F \colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}_{\Delta}^{\mathbf{ms}}$$

and yields as output a 2-Cartesian fibration $\chi_{\mathbb{C}}(F) \to N^{\mathrm{sc}}(\mathbb{C})$. In form, this relative nerve will actually seem slightly *more* complicated than the associated straightening functor. However, it will enable us to more easily make the comparison with the strict 2-categorical relative nerve construction of [7]. The particular virtue of our relative 2-nerve construction in this regard is that, given a strict 2-functor

$$F: \mathbb{C}^{\mathrm{op}} \longrightarrow 2\mathrm{Cat}.$$

we can compute the relative 2-nerve in terms of strict 2-functors into \mathbb{C} and F(x), without first passing to simplicial sets.

In our previous papers [3] and [2], we defined two variants of the relative 2-nerve, which provided ∞ -bicategories fibred in $(\infty, 1)$ -categories. In this section, we will upgrade the later of these constructions to provide the desired $\chi_{\mathbb{C}}$.

Remark A.1. Our choice of notation $\chi_{\mathbb{C}}$ for the relative 2-nerve of a functor $F: \mathbb{C}^{\mathrm{op}} \to \mathrm{Set}^{\mathbf{ms}}_{\Delta}$ does in fact collide with the choice of notation in [3] and [2]. An ideal choice of notation would involved a superscript $\chi_{\mathbb{C}}^{\varepsilon}$ where ε denotes one of the four variances for bicategorical fibrations. We will use this rather abusive notation to improve readibility since we will only consider the *outer Cartesian* variance.

Definition A.2. Given a totally ordered set I, the 2-category $\mathbb{O}_{i^{\star}}^{I}$ has

- Objects given by subsets $S \subseteq I$ such that $\min(S) = i$.
- Each mapping category $\mathbb{O}^I_{i\uparrow}(S,T)$ is a poset whose objects $\mathcal{U}\colon S\longrightarrow T$ are given by subsets $\mathcal{U}\subseteq I$ such that

$$\min(\mathcal{U}) = \max(S), \ \max(\mathcal{U}) = \max(T), \ S \cup \mathcal{U} \subseteq T,$$

ordered by inclusion.

• Composition is given by union.

These lax slice categories piece together into a 2-functor

$$(\mathbb{O}^I)^{\mathrm{op}} \longrightarrow 2\mathrm{Cat}$$
 $i \longmapsto \mathbb{O}^I_{i\uparrow}$

so that, in particular for any $J \subset I$ with $i = \min(I)$ and $j = \min(J)$, we have 2-functors

$$\omega_{I,J} \colon \mathbb{O}^I(i,j) \times \mathbb{O}^J_{j\uparrow} \longrightarrow \mathbb{O}^I_{i\uparrow}$$

given on objects by the union of sets. It is an easy check that these functors are injective on objects, 1-morphisms, and 2-morphisms.

The 2-categories $\mathbb{O}^I_{i \uparrow}$ play a central role in our relative nerve construction.

Construction A.3. Let

$$F \colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}^{\mathbf{ms}}_{\Lambda}$$

be a $\operatorname{Set}_{\Delta}^+$ -enriched functor. We define a marked-biscaled simplicial set $\chi_{\mathbb{C}}(F)$ as follows. An n-simplex $\Delta^n \to \chi_{\mathbb{C}}(F)$ consists of

- A simplex $\sigma: \Delta_{\flat}^n \to \mathcal{N}^{\mathrm{sc}}(\mathbb{C})$.
- For every $\emptyset \neq I \subset [n]$ with $\min(I) = i$, a map of marked-scaled simplicial sets

$$\theta_I \colon \operatorname{N^{ms}}(\mathbb{O}^I_{i \nearrow})^{\flat} \longrightarrow F(\sigma(i))$$

such that, for every $\emptyset \neq J \subset I \subset [n]$ with $\min(J) = j$ and $\min(i) = i$, the diagram

$$\begin{split} \mathbf{N}^{\mathrm{ms}}(\mathbb{O}^{I}(i,j)) \times \mathbf{N}^{\mathrm{ms}}(\mathbb{O}^{J}_{j\uparrow}) & \longrightarrow \mathbf{N}^{\mathrm{ms}}(\mathbb{O}^{I}_{i\uparrow}) \\ \mathbf{N}^{(\sigma) \times \theta_{J}} \Big\downarrow & \qquad \qquad \downarrow \theta_{I} \\ \mathbf{N}^{\mathrm{ms}}(\mathbb{C}(\sigma(i),\sigma(j))) \times F(\sigma(j)) & \xrightarrow{F(-)} F(\sigma(i)) \end{split}$$

commutes.

We then define markings and scalings on $\rho_{\mathbb{C}}(F)$.

• A 1-simplex $\Delta^1 \to \chi_{\mathbb{C}}(F)$ is marked if the corresponding map $\theta_{[1]}: \mathcal{N}^{\mathrm{ms}}(\mathbb{O}^1_{0^{\uparrow}}) \to F(\sigma(i))$ descends to a map

$$N^{ms}(\mathbb{O}^1_{0\nearrow})^{\sharp} \longrightarrow F(\sigma(0)).$$

• A 2-simplex $\Delta^2 \to \chi_{\mathbb{C}}(F)$ is lean if the corresponding map

$$N^{ms}(\mathbb{O}^2_{07}) \longrightarrow F(\sigma(0))$$

descends to a map

$$N^{ms}(\mathbb{O}^2_{0\uparrow})_{\sharp} \longrightarrow F(\sigma(0)).$$

• A 2-simplex $\Delta^2 \to \chi_{\mathbb{C}}(F)$ is thin if and only if it is lean and the corresponding 2-simplex $\sigma: \Delta^2 \to \mathcal{N}^{\mathrm{sc}}(\mathbb{C})$ is thin.

Note that there is a canonical forgetful functor

$$\chi_{\mathbb{C}}(F) \longrightarrow N^{\mathrm{sc}}(\mathbb{C})$$
 $(\sigma, \{\theta_I\}) \longmapsto \sigma$

which sends thin triangles to thin triangles.

Definition A.4. The relative bicategorical nerve over a 2-category \mathbb{C} is the functor

$$\chi_{\mathbb{C}} \colon \left(\operatorname{Set}_{\Delta}^{\mathbf{ms}} \right)^{\mathbb{C}^{\operatorname{op}}} \longrightarrow \left(\operatorname{Set}_{\Delta}^{\mathbf{mb}} \right)_{/\operatorname{N}^{\operatorname{sc}}(\mathbb{C})}$$

$$F \longmapsto \chi_{\mathbb{C}}(F)$$

By the adjoint functor theorem, XC admits a left adjoint, which we will denote by

$$\phi_{\mathbb{C}} \colon (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\operatorname{N}^{\operatorname{sc}}(\mathbb{C})} \longrightarrow (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathbb{C}^{\operatorname{op}}} \ .$$

Lemma A.5. The functor $\chi_{\mathbb{C}}$ preserves trivial fibrations.

Proof. We need only check that the lifting problems

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{CC}(F) \\ f & & & \mathrm{CC}(\mu) \\ B & \longrightarrow & \mathrm{CC}(G) \end{array}$$

have solutions when $\mu: F \Rightarrow G$ is a projective (pointwise) trivial fibration and $f: A \to B$ is a generating cofibration of marked-biscaled simplicial sets. The proof is virtually identical to the proof of [2, Prop. 3.0.11].

Corollary A.6. The functor $\phi_{\mathbb{C}}$ preserves cofibrations.

A.1. Identifying $\phi_{\mathbb{O}^n}$

Let $\mathbb C$ be a 2-category. Then we can define a 2-functor

$$\mathbb{C}^{\mathrm{op}} \longrightarrow 2\mathrm{Cat}, \ c \longmapsto \mathbb{C}_{c^{\uparrow}}$$

that maps a 1-morphism $f: c \to d$ to the functor $f^*: \mathbb{C}_{d\uparrow} \to \mathbb{C}_{c\uparrow}$ given by precomposition with f. It easy to verify that given a 2-morphism $\alpha: f \Rightarrow g$ we can construct a natural transformation $f^* \Rightarrow g^*$ whose component at an object $u: d \to x$ is given by $\alpha * u$. Passing to $\operatorname{Set}_{\Delta}^+$ -enriched categories we thus obtain, for any strict 2-category \mathbb{C} , a $\operatorname{Set}_{\Delta}^+$ -enriched functor

$$\mathbb{C}_{-\uparrow} \colon \mathbb{C}^{\mathrm{op}} \longrightarrow \mathrm{Set}_{\Delta}^{\mathbf{ms}}, \ c \longmapsto \mathrm{N}^{\mathrm{ms}}(\mathbb{C}_{c^{\uparrow}})$$

Definition A.7. In the particular case where $\mathbb{C} = \mathbb{O}^n$, we will denote the functor constructed above by

$$\mathfrak{O}^n \colon (\mathbb{O}^n)^{\mathrm{op}} \longrightarrow \mathrm{Set}_{\Delta}^{\mathbf{ms}}$$

Notation. The canonical normal lax functor $\xi_n : [n] \to \mathbb{O}^n$ gives rise to an inclusion of scaled simplicial sets which we denote by

$$p_n : \Delta_{\mathsf{b}}^n \longrightarrow \mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n).$$

We will equip Δ^n with the minimal marking and lean scaling, and conventionally view p_n as an object in $\left(\operatorname{Set}_{\Delta}^{\mathbf{mb}}\right)_{/\operatorname{N}^{\operatorname{sc}}(\mathbb{O}^n)}$.

Lemma A.8. Let $F:(\mathbb{O}^n)^{\mathrm{op}} \to \mathrm{Set}_{\Delta}^{\mathbf{ms}}$ be a Set_{Δ}^+ -enriched functor. There is a natural bijection

$$\operatorname{Nat}_{\mathbb{C}^{\operatorname{op}}}(\mathfrak{O}^n, F) \cong \operatorname{Hom}_{\left(\operatorname{Set}_{\Delta}^{\mathbf{mb}}\right)_{/\operatorname{Nsc}(\mathbb{O}^n)}}(p_n, \chi_{\mathbb{O}}(F)).$$

Consequently, we have an equivalence of $\operatorname{Set}_{\Lambda}^+$ -enriched functors $\phi_{\mathbb{O}^n}(p_n) \cong \mathfrak{O}^n$.

Proof. Follows immediately from unwinding the definitions.

Corollary A.9. Denote by

$$p_1^{\sharp} : (\Delta^1)^{\sharp} \longrightarrow \mathcal{N}^{\mathrm{sc}}(\mathbb{O}^1),$$

$$(p_2)_{\flat \subset \sharp} : (\Delta^2)^{\flat}_{\flat \subset \sharp} \longrightarrow \mathcal{N}^{\mathrm{sc}}(\mathbb{O}^2), \ and$$

$$(p_2)_{\sharp} : (\Delta^2)^{\flat}_{\sharp} \longrightarrow \mathcal{N}^{\mathrm{sc}}(\mathbb{O}^2)_{\sharp}$$

the obvious decorated versions of the p_n . Then

$$\phi_{\mathbb{O}^{1}}(p_{1}^{\sharp}) \colon (\mathbb{O}^{1})^{\mathrm{op}} \longrightarrow (\operatorname{Set}_{\Delta}^{\mathbf{ms}}), \ i \longmapsto \operatorname{N}^{\mathrm{ms}}(\mathbb{O}_{i\uparrow}^{1})^{\sharp}$$

$$\phi_{\mathbb{O}^{2}}((p_{2})_{\flat \subset \sharp}) \colon (\mathbb{O}^{2})^{\mathrm{op}} \longrightarrow (\operatorname{Set}_{\Delta}^{\mathbf{ms}}), \ i \longmapsto \operatorname{N}^{\mathrm{ms}}(\mathbb{O}_{i\uparrow}^{2})_{\sharp}, \ and$$

$$\phi_{\mathbb{O}^{2}}((p_{2})_{\flat \subset \sharp}) \colon (\mathbb{O}^{2})_{\sharp}^{\mathrm{op}} \longrightarrow (\operatorname{Set}_{\Delta}^{\mathbf{ms}}), \ i \longmapsto \operatorname{N}^{\mathrm{ms}}(\mathbb{O}_{i\uparrow}^{2})_{\sharp}^{\dagger}$$

where \dagger denotes the marking in which the unique morphism $02 \rightarrow 012$ is marked.

Proof. All identifications except the last are immediate from the definitions. The additional marking in the final case follows from the necessity that the functor have source $\mathbb{O}^2_{\mathfrak{t}}$.

Notation. We will denote the three functors above by $(\mathfrak{O}^1)^{\sharp}$, $(\mathfrak{O}^2)_{\flat \subset \sharp}$, and $(\mathfrak{O}_2)_{\sharp}$, respectively.

A.2. Identifying $\mathbb{S}t_{\mathbb{O}^n}$

Our comparison will be with a very specific version of the straightening functor:

Notation. For a 2-category \mathbb{C} , we view \mathbb{C} as a $\operatorname{Set}_{\Delta}^+$ -enriched category. The counit $\varepsilon_{\mathbb{C}}: \mathfrak{C}^{\operatorname{sc}}(\operatorname{N}^{\operatorname{sc}}(\mathbb{C})) \to \mathbb{C}$ is an equivalence of $\operatorname{Set}_{\Delta}^+$ -enriched categories. We will denote by

$$\mathbb{S}t_{\mathbb{C}} \colon (\mathrm{Set}_{\Delta}^{\mathbf{mb}})_{/N^{\mathrm{sc}}(\mathbb{C})} \longrightarrow (\mathrm{Set}_{\Delta}^{\mathbf{ms}})^{\mathbb{C}^{\mathrm{op}}}$$

the relative straightening functor $\mathbb{S}t_{\varepsilon_{\mathbb{C}}}$.

We now unravel the definitions to characterize $\mathbb{S}t_{\mathbb{O}^n}(p_n)$. By construction, the underlying functor to $\operatorname{Set}_{\Delta}^+$ is given by the Yoneda embedding on the $\operatorname{Set}_{\Delta}^+$ -enriched category

$$\mathbb{O}^n \coprod_{\mathfrak{C}^{\mathrm{sc}}(\mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n))} \mathfrak{C}^{\mathrm{sc}}(\mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n)) \coprod_{\mathfrak{C}^{\mathrm{sc}}(\Delta^n_{\iota})} \mathfrak{C}^{\mathrm{sc}}((\Delta^n_{\flat})^{\triangleright})$$

We note that $\mathbb{O}^n = \mathfrak{C}^{\mathrm{sc}}(\Delta^n_{\flat})$, and by the triangle identities for the adjunction $\mathfrak{C}^{\mathrm{sc}} \dashv \mathrm{N}^{\mathrm{sc}}$, we see that the induced map

$$\mathfrak{C}^{\mathrm{sc}}(\Delta^n_{\flat}) \to \mathbb{O}^n$$

is simply the identity. The pushout above thus collapses to simply $\mathfrak{C}^{\mathrm{sc}}((\Delta_{\flat}^n)^{\triangleright})$. We can then describe the marked-scaled simplicial set $\mathbb{S}_{\mathfrak{t}_{\mathbb{O}^n}}(\Delta_{\flat}^n)(i)$ as the poset $\mathcal{L}_{\flat}^n(i)$ described in Definition 3.11. To ease the notation let us denote $\mathcal{L}_{\flat}^n(i)$ simply by \mathcal{L}_i^n .

Construction A.10. We construct a morphism of marked-scaled simplial sets $\eta_i^n : \mathcal{L}_i^n \to \mathcal{N}^{\mathrm{ms}}(\mathbb{O}_{i\uparrow}^n)$ whose underlying map of simplicial sets is given by (the nerve of) a normal lax functor defined as follows:

- On objects, $S \mapsto S$.
- On morphisms $S \subset T$ is sent to the mophism $\{\max(S), \max(T)\}: S \to T$.

The fact that, for $S \subset T \subset V$, we have $\{\max(S), \max(V)\} \subset \{\max(S), \max(T), \max(V)\}$ gives us our compositors. The fact that if S = T, we have $\{\max(S), \max(T)\} = \{\max(S)\}$ gives strict unitality. Since both marked-scaled simplicial sets carry the minimal marking we only need to check that η_i^n preserves the scaling. Let $S_0 \subset S_1 \subset S_2$ be a 2-simplex in the source. If there are i, j such that $\max(S_i) = \max(S_j)$, then it follows immediately that $\{\max(S_0), \max(S_1), \max(S_2)\} = \{\max(S_0), \max(S_2)\}$.

The following lemma follows immediately from our definitions.

Lemma A.11. The maps η_i^n define natural transformations of $\operatorname{Set}_{\Delta}^+$ -enriched functors $\eta^n : \operatorname{St}_{\mathbb{O}^n}(\Delta_{\flat}^n) \to \mathfrak{D}^n$.

Proposition A.12. The morphisms $\eta_i^n : \operatorname{St}_{\mathbb{O}^n}(\Delta_{\flat}^n)(i) \to \mathfrak{O}^n(i)$ are equivalences of marked-scaled simplicial sets.

We will prove this proposition in a series of lemmata. Since both simplicial sets are equipped with the minimal marking, it suffices to show that the map is an equivalence on underlying scaled simplicial sets by Theorem 2.51. Since $\mathfrak{O}^n(i) = \mathrm{N}^{\mathrm{sc}}(\mathbb{O}^n_{i\nearrow})$, it suffices to show that the induced map

$$\xi_i^n \colon \mathfrak{C}^{\mathrm{sc}}[(\mathbb{S}\mathbf{t}_{\mathbb{O}^n}(\Delta^n_{\flat}))(i)] \longrightarrow \mathbb{O}^n_{i \nearrow i}$$

is an equivalence of $\operatorname{Set}_{\Delta}^+$ -enriched categories. Since this map is clearly bijective on objects, it suffices to check that the induced morphisms on mapping spaces are equivalences.

In both cases, the mapping spaces are nerves of posets.

- For $S,T \in \mathbb{O}^n_{i\uparrow}$, the mapping space $\mathfrak{O}^n_i(S,T)$ is the poset of chains $U \subset [n]$ such that $\min(U) = \max(S), \, \max(U) = \max(T), \, \text{and} \, S \cup U \subset T$. Equivalently, this is the poset $\mathbb{O}^T(\max(S), \max(T))$, equipped with the minimal marking.
- For $S, T \in Q_i^n$, the mapping space $\mathfrak{C}^{\mathrm{sc}}[(\mathfrak{St}_{\mathbb{O}^n}(\Delta_b^n))(i)](S,T)$ is the poset of chains

$$S \subset S_1 \subset \cdots \subset S_k \subset T$$

in \mathcal{L}_i^n . An inclusion $\vec{S} \subset \vec{U}$ is marked if and only if, for every $S_i \subset S_{i+1}$ in \vec{S} , every $U_\ell \in \vec{U}$

between S_i and S_{i+1} , either $\max(U_\ell) = \max(S_i)$ or $\max(U_\ell) = \max(S_{i+1})$. Notice that $T \setminus S$ could have elements lower than $\max(S)$.

The map

$$\xi_i^n: \mathfrak{C}^{\mathrm{sc}}[(\mathbb{S}\mathfrak{t}_{\mathbb{O}^n}(\Delta_{\flat}^n))(i)](S,T) \to \mathfrak{O}_i^n(S,T)$$

sends a chain $S \subset S_1 \subset \cdots \subset S_k \subset T$ to the chain

$$\xi_i^n(\vec{S}) = \bigcup_{V \in \vec{S}} \{ \max(V) \}.$$

Definition A.13. For ease of notation, we define

$$\mathbb{L}^n_i := \mathfrak{C}^{\mathrm{sc}}[(\mathbb{S}\mathrm{t}_{\mathbb{O}^n}(\Delta^n_{\flat}))(i)]$$

For any $S, T \in \mathbb{L}_i^n$, we define a full subposet $\mathbb{B}_i^n(S,T) \subset \mathbb{L}_i^n(S,T)$ consisting of chains

$$S \subset [T, s_1] \subset \cdots \subset [T, s_k] \subset T$$

where we define for every $s \in T$ the subset $[T, s] = \{r \in T \mid r \leq s\}$.

Lemma A.14. An inclusion $\vec{S} \subset \vec{U}$ represents a marked morphism in $\mathbb{L}^n_i(S,T)$ if and only if its image under ξ^n_i is degenerate.

Proof. Immediate from the definition.

Lemma A.15. The restriction of the map ξ_i^n

$$\xi_i^n \colon \mathbb{B}_i^n(S,T) \longrightarrow \mathbb{O}^T(\max(S), \max(T))$$

is an equivalence of marked simplicial sets.

Proof. We define a map

$$\gamma \colon \mathbb{O}^T(\max(S), \max(T)) \longrightarrow \mathbb{B}_i^n(S, T)$$

which sends $\max(S) < s_1 < \cdots < s_k < \max(T)$ to the chain

$$S \subset [T, s_1] \subset \cdots \subset [T, s_k] \subset T$$
.

We then note that $\xi_i^n \circ \gamma = \text{id}$. We claim that $\gamma \circ \xi_i^n \leqslant \text{id}$, which yields a marked homotopy $\gamma \circ \xi_i^n$ to id. To prove the claim we note that $\gamma \circ \xi_i^n(\vec{S})$ is given by \vec{S} if $S_1 \neq [T, s_0]$ with $s_0 = \max(S)$ or by $\vec{S} \setminus [T, s_0]$ in which case the existence of the marked morphism $\gamma \circ \xi_i^n(\vec{S}) \to \vec{S}$ follows immediately. \square

Lemma A.16. The inclusion $\iota: \mathbb{B}_i^n(S,T) \to \mathbb{L}_i^n(S,T)$ is an equivalence of marked simplicial sets.

Proof. Let $s_j \in T$. We define $\mathbb{L}^s \subset \mathbb{L}$ as the full subposet consisting of those chains

$$\vec{S} = S \subset S_1 \subset \cdots \subset S_k \subset T$$
,

such that $S_i = [T, s_i]$ whenever $s_i \ge s_j$. Note that if $s_j \le s_0 = \max(S)$ then it follows that $\mathbb{L}^s = \mathbb{B}$. Let $T_S = \{s_j \in T \mid s_j \ge s_0\}$ and consider a filtration

$$\mathbb{B} = \mathbb{L}^{s_0} \subset \mathbb{L}^{s_1} \subset \cdots \subset \mathbb{L}^{s_m} \subset \mathbb{L}^{s_m+1} = \mathbb{L}, \text{ with } s_m = \max(T)$$

Our goal is to show that each step in the filtration is a weak equivalence of marked simplicial sets. We denote by $\iota_j: \mathbb{L}^j \to \mathbb{L}^{j+1}$ for $j = 0, \ldots, s_m$. Let $\vec{S} = S \subset S_1 \subset \cdots \subset S_k \subset T$ be an object of \mathbb{L}^{j+1} we construct a new chain $\pi_j(\vec{S})$ by replacing each S_ℓ with $s_\ell \geqslant s_j$ with its corresponding

 $[T, s_{\ell}]$. This definition yields a functor

$$\pi_j \colon \mathbb{L}^{j+1} \longrightarrow \mathbb{L}^j, \ \vec{S} \longmapsto \pi_j(\vec{S})$$

such that $\pi_j \circ \iota_j = \text{id.}$ Let $\zeta_j = \iota_j \circ \pi_j$. We construct a functor

$$\theta_i \colon \mathbb{L}^{j+1} \longrightarrow \mathbb{L}^{j+1}$$

that appends to each chain $\vec{S} \in \mathbb{L}^{j+1}$ the object $[T, s_j]$ if there exists some $S_\ell \in \vec{S}$ such that $\max(S_\ell) = s_j$ or leaves the chain untouched otherwise. Note that if $s_j = s_m$ then this functor is the identity. We also observe that we have a natural transformation id $\leqslant \theta_j$ and $\zeta_j \leqslant \theta_j$ whose components are marked. It follows that each ι_j is a weak equivalence and consequently so is ι .

Proof (of Proposition A.12). We simply apply Lemma A.16, Lemma A.15, and 2-out-of-3. \Box

Turning now to the cases $(p_1)^{\sharp}$, $(p_2)_{\flat \subset \sharp}$, and $(p_2)_{\sharp}$, we see that the corresponding straightenings are obtained from $\operatorname{St}_{\mathbb{O}^1}(p_1)$ and $\operatorname{St}_{\mathbb{O}^2}(p_2)$ by maximally marking or maximally scaling the values of the functors, respectively. We then have the following

Corollary A.17. The transformations ξ^n , n=1,2 induce equivalences of enriched functors

$$(\xi^{1})^{\sharp} \colon \operatorname{St}_{\mathbb{O}^{1}}(p_{1}^{\sharp}) \longrightarrow (\mathfrak{O}^{1})^{\sharp},$$

$$(\xi^{2})_{\flat \subset \sharp} \colon \operatorname{St}_{\mathbb{O}^{2}}((p_{2})_{\flat \subset \sharp}) \longrightarrow (\mathfrak{O}^{2})_{\flat \subset \sharp}, \ and$$

$$(\xi^{2})_{\sharp} \colon \operatorname{St}_{\mathbb{O}^{2}_{\sharp}}((p_{2})_{\sharp}) \longrightarrow (\mathfrak{O}^{2})_{\sharp}$$

Proof. The morphism $(\xi^1)^{\sharp}$ is an isomorphism, and it is a quick check to extend the previous arguments to cover the case $(\xi^2)_{\flat \subset \sharp}$. One then notes that, for each $i \in \mathbb{O}^2$, the *i*-component of $(\xi^2)_{\sharp}$ is a pushout of the *i*-component of $(\xi^2)_{\flat \subset \sharp}$ along the inclusion $(\Delta^1)^{\flat} \to (\Delta^1)^{\sharp}$, and thus is an equivalence.

Remark A.18. As in [3, Prop. 4.1.1] any 2-functor $f: \mathbb{C} \to \mathbb{D}$ yields diagrams

$$\begin{array}{ccc} (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathbb{D}^{\operatorname{op}}} & \xrightarrow{f^*} & (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathbb{C}^{\operatorname{op}}} \\ & & & & \downarrow \mathbb{X}\mathbb{C} \end{array}$$

$$(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\operatorname{N^{\operatorname{sc}}}(\mathbb{D})} \xrightarrow{\operatorname{N^{\operatorname{sc}}}(f)^*} (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\operatorname{N^{\operatorname{sc}}}(\mathbb{C})}$$

and

$$(\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathbb{D}^{\operatorname{op}}} \xleftarrow{f_!} (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathbb{C}^{\operatorname{op}}}$$

$$\downarrow_{\mathbb{D}} \qquad \qquad \uparrow_{\mathbb{C}} \\ (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\operatorname{N^{\operatorname{sc}}}(\mathbb{D})} \underset{\operatorname{N^{\operatorname{sc}}}(f)_!}{\not\sim} (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\operatorname{N^{\operatorname{sc}}}(\mathbb{C})}$$

which commute up to natural isomorphism.

Theorem A.19. There exists a unique family of natural weak equivalences

$$\xi^{\mathbb{C}}(X) \colon \mathbb{S} \mathrm{t}_{\mathbb{C}} \Longrightarrow \phi_{\mathbb{C}}$$

indexed by pairs (\mathbb{C}, X) consisting of a 2-category \mathbb{C} and $X \in (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\operatorname{N}^{\operatorname{sc}}(\mathbb{C})}$ with the following properties.

- 1. On the maps p_n for $n \ge 0$, p_1^{\sharp} , $(p_2)_{\flat \subset \sharp}$, and $(p_2)_{\sharp}$, the transformations $\xi^{\mathbb{C}}(X)$ coincide with the transformations ξ^n from Proposition A.12 and Corollary A.17.
- 2. For every map $g: X \to Y$ in $(\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{\operatorname{N}^{\operatorname{sc}}(\mathbb{C})}$, the diagram

$$\begin{array}{ccc} \operatorname{St}_{\mathbb{C}}(X) & \xrightarrow{\xi^{\mathbb{C}}(X)} \ \phi_{\mathbb{C}}(X) \\ \operatorname{St}_{\mathbb{C}}(g) \Big\downarrow & & & & & & & & & \\ \operatorname{St}_{\mathbb{C}}(Y) & & & & & & & & \\ \end{array}$$

commutes

3. For every 2-functor $f: \mathbb{C} \to \mathbb{D}$, the diagram

$$f_{!} \operatorname{St}_{\mathbb{C}}(X) \xrightarrow{f_{!} \xi^{\mathbb{C}}(X)} f_{!} \phi_{\mathbb{C}}(X)$$

$$\downarrow \cong \qquad \qquad \downarrow \cong$$

$$\operatorname{St}_{\mathbb{D}}(f_{!}X) \xrightarrow{\eta_{\mathbb{C}}(f_{!}X)} \phi_{\mathbb{C}}(f_{!}X)$$

commutes.

Proof. This is identical to the proofs of [3, Prop 4.3.1 and 4.3.1].

Corollary A.20. The adjunction

$$\phi_{\mathbb{C}}: (\operatorname{Set}_{\Delta}^{\mathbf{mb}})_{/\operatorname{N}^{\operatorname{sc}}(\mathbb{C})} \longleftrightarrow (\operatorname{Set}_{\Delta}^{\mathbf{ms}})^{\mathbb{C}^{\operatorname{op}}}: \chi_{\mathbb{C}}$$

is a Quillen equivalence.

Proof. Since $\phi_{\mathbb{C}}$ preserves cofibrations, and is naturally weakly equivalent to $St_{\mathbb{C}}$, it preserves trivial cofibrations, and thus is left Quillen. Moreover, the left-derived functors of $St_{\mathbb{C}}$ and $\phi_{\mathbb{C}}$ agree, and the former is an equivalence.

A.3. Comparison to the strict case

We now establish a comparison result with the strict 2-categorical case, as worked out by Buckley in [7]. We will heavily leverage two facts to ease the proof of this comparison results

- For a strict 2-functor $F: \mathbb{C}^{\mathrm{op}} \to 2\mathrm{Cat}$, we can describe $\chi_{\mathbb{C}}(F)$ entirely in terms of 2-functors into \mathbb{C} and F(x), for $x \in \mathbb{C}$.
- The Duskin 2-nerve $N_2(\mathbb{C})$ of any strict 2-category \mathbb{C} is 3-coskeletal.

Making use of these two facts allows us to construct a comparison map by checking a finite number of cases by hand. Once the comparison is established, we can work with strict 2-categories to prove that it is a fibre-wise equivalence.

Let us now introduce the 2-categorical Grothendieck construction we wish to compare with. Appropriately dualizing Buckley's construction⁶, the *strict 2-categorical Grothendieck construction* of a 2-functor

$$F \colon \mathbb{C}^{\mathrm{op}} \longrightarrow 2\mathrm{Cat}$$

is the 2-category El(F) which has

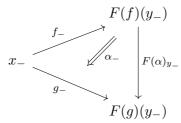
⁶Buckley defines a construction that takes as input a functor $F: \mathbb{C}^{(\text{op,op})} \to 2\text{Cat}$ where both 1- and 2-morphisms have been reversed.

- Objects: pairs (x, x_{-}) with $x \in \mathbb{C}$ and $x_{-} \in F(x)$.
- Morphisms:

$$(f, f_-): (x, x_-) \longrightarrow (y, y_-)$$

where $f: x \to y$ in \mathbb{C} , and $f_-: x_- \to F(f)(y_-)$.

• 2-Morphisms: $(\alpha, \alpha_{-}): (f, f_{-}) \Rightarrow (g, g_{-})$, where $\alpha: f \Rightarrow g$ is a 2-morphism in \mathbb{C} , and α_{-} fits in the diagram



The resulting functor $\mathrm{El}(F) \to \mathbb{C}$ is a 2-Cartesian fibration, where

- (f, f_{-}) is Cartesian if f_{-} is an equivalence, and
- (α, α_{-}) is coCartesian if α_{-} is an isomorphism.

Our aim is to prove the following

Theorem A.21. Let

$$F: \mathbb{C}^{(\text{op},-)} \longrightarrow 2\text{Cat}$$

be a 2-functor, and let \tilde{F} denote the composite

$$\mathbb{C}^{(op,-)} \longrightarrow 2Cat \longrightarrow Set^{\mathbf{ms}}_{\Delta}$$

Then there is an equivalence

$$(\mathrm{N}_2(\mathrm{El}(F)), M, T \subset C) \longleftarrow^{\simeq} \mathbb{X}\mathbb{C}(\tilde{F})$$

$$N^{\mathrm{sc}}(\mathbb{C})$$

of 2-Cartesian fibrations over $N^{sc}(\mathbb{C})$.

We begin by showing there is a map in one direction. For ease of notation, given a morphism $\phi: x \to y$ in \mathbb{C} , we will write $\phi^* := F(\phi)$. We will employ the same convention for 2-morphisms. Since the 2-nerve of a 2-category is 3-coskeletal, it suffices to define a map

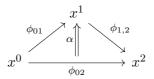
$$\operatorname{sk}_3(\chi_{\mathbb{C}}(F)) \longrightarrow N_2(\operatorname{El}(F))$$

which is compatible with markings and scalings.

On 0- and 1-simplices, the data specified by the simplices in both constructions is identical. A 2-simplex in $\chi_{\mathbb{C}}(F)$ consists of the following data:

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• A 2-simplex



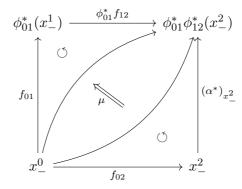
• Three 1-simplices

$$f_{12} \colon x_{-}^{1} \longrightarrow \phi_{12}^{*}(x_{-}^{2})$$
$$f_{01} \colon x_{-}^{0} \longrightarrow \phi_{01}^{*}(x_{-}^{1})$$

and

$$f_{02} \colon x_-^0 \longrightarrow \phi_{02}^*(x_-^2)$$

• A diagram



in \mathbb{C} .

These data are identical to the data of a 2-simplex in El(F). It is immediate that markings and scalings coincide under these correspondences.

Finally, we note that 3-simplices in El(F) are simply compatibility conditions on 2-morphisms. It is a long but easy check to see that, given a 3-simplex in $\chi_{\mathbb{C}}(F)$, the corresponding 2-simplices in El(F) are compatible. We have thus shown

Proposition A.22. There is a morphism of 2-Cartesian fibrations

$$\tau: \chi_{\mathbb{C}}(F) \longrightarrow \mathrm{El}(F).$$

The final ingredient in our proof will involve a comparsion of 2-functors.

Definition A.23. We denote by $\pi^n: \mathbb{O}^n_{0\nearrow} \to \mathbb{O}^n$ the canonical projection. This sends $S \mapsto \max(S)$, and sends a morphism $\mathcal{U}: S \to T$ to the set \mathcal{U} .

Given a 2-category C, we call a 2-functor

$$f:\mathbb{O}^n_{0,7}\to\mathbb{C}$$

peripatetically constant if it sends every morphism $\mathcal{U}:S\to T$ where $\max(S)=\max(T)$ to an identity, and every 2-morphism between such morphisms to an identity as well. We denote the set of peripatetically constant functors $\mathbb{O}^n_{0\uparrow}\to\mathbb{C}$ by

$$\operatorname{Hom}^{\operatorname{PC}}(\mathbb{O}^n_{0\nearrow},\mathbb{C}).$$

Lemma A.24. For any $n \ge 0$, the 2-functor $\pi^n : \mathbb{O}_{0,\uparrow}^n \to \mathbb{O}^n$ induces a bijection

$$(\pi^n)^*: \mathrm{Hom}(\mathbb{O}^n, \mathbb{C}) \stackrel{\cong}{\longrightarrow} \mathrm{Hom}^{\mathrm{PC}}(\mathbb{O}^n_{0^{\not r}}, \mathbb{C})$$

Proof. We can define a strict 2-functor

$$s^n \colon \mathbb{O}^n \longrightarrow \mathbb{O}^n_{0 \nearrow}$$
$$j \longmapsto [0, j]$$

which acts as the identity on 1- and 2-morphisms. Since $\pi^n \circ s^n = \mathrm{id}$, we have $(s^n)^* \circ (\pi^n)^* = \mathrm{id}$, and thus, π^n_* is injective. It is immediate from unraveling the definitions that the image of $(\pi^n)^*$ is precisely the peripatetically constant functors.

Corollary A.25. Let $\mathbb D$ be a 2-category, and denote by * the terminal 2-category. There is an isomorphism

$$\chi_*(\mathbb{D}) \cong \mathrm{N}^{\mathrm{sc}}(\mathbb{D}).$$

Proof. An *n*-simplex in $\chi_*(\mathbb{D})$ consists of 2-functors

$$\theta_I \colon \mathbb{O}_{i^{\nearrow}}^I \longrightarrow \mathbb{D}$$

for every non-empty $I \subset [n]$, such that for every $J \subset I \subset [n]$, the diagram

$$\mathbb{O}^{I}(i,j) \times \mathbb{O}^{J}_{j\uparrow} \longrightarrow \mathbb{O}^{I}_{i\uparrow}$$

$$\downarrow \qquad \qquad \downarrow$$

$$* \times \mathbb{D} \longrightarrow \mathbb{D}$$

commute. Such functors are uniquely determined by the map $\theta_{[n]}: \mathbb{O}^n_{0\uparrow} \to \mathbb{D}$, and the commutativity of the diagrams above is equivalent to requiring that $\theta_{[n]}$ be peripatetically constant.

Consequently, we obtain a bijection on sets of n-simplices $N^{sc}(\mathbb{D})_n \cong (\chi_*(\mathbb{D}))_n$ by pulling back along π^n . The corollary follows from checking directly that these bijections respect face and degeneracy maps.

Proof (of Theorem A.21). By [4, Proposition 3.35], it will suffice for us to show that this morphism is an equivalence on fibres. By construction, the fibre of El(F) over $x \in \mathbb{C}$ is precisely the 2-category F(x), and the fibre of $\chi_{\mathbb{C}}(F)$ over x is also precisely $\chi_{x}(F(x)) \cong \mathbb{N}^{sc}(F(x))$.

It is a quick explicit check that, on 0-,1-,2-, and 3-simplices, the map $\tau: \chi_x(F(c)) \to \operatorname{El}_x(F(x)) \cong \operatorname{N}^{\operatorname{sc}}(F(x))$ agrees with the isomorphism of Corollary A.25. The theorem then follows from 3-coskeletalness.

Index of Notation

For ease of reference, we provide here a non-comprehensive list of notations used in this paper.

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